Algebra

Jiwu Jang

June 17, 2023

This is a note on a short talk about Noetherian rings, Artinian rings, and short exact sequences (SES), given by Mustafa Nawaz.

Definition 0.1. A ring R is Noetherian if it satisfies the following equivalent conditions of the proposition:

- Every set of ideals $S \neq \emptyset$ in R has a maximal element.
- Every ascending chain of ideals becomes stationary.
- Every ideal in R is finitely generated.

Proof of $(1 \implies 2)$. The set $(x_m)_{m=1}^n$ has a maximal element, say (x_n). □

Proof of $(2 \implies 1)$. Assume it's false. Then, there exists a nonempty subset T of Σ (which is the chain of ideals) with no maximal element, we construct inductively. □

Proof of $(2 \implies 3)$. Huh? □

Proof of $(3 \implies 2)$. Let $R_1 \subset R_2 \subset \dots$ be an ascending chain of ideals. Then $I = \bigcup_{n=1}^{k} R_n$ is a
https://math.stackexchange.com/questions/3912632/checking-the-proof-that-if-every-ideal-of-r-is-finitely-generated-then-r-is □

Exercise 0.2. Prove it.

- Every PID is Noetherian
- Every field is Noetherian

Theorem 0.3 (Hilbert's basis theorem (Nullstellensatz))
R is Noetherian $\iff$ $R[x_1, \dots, x_n]$ is Noetherian.

Claim — R is Noetherian $\implies$ $R[x]$ is Noetherian.

Proof. Let I be an ideal of $R[x]$. Let I_k be the ideal of leading coefficient of degree k elements. Then $a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \in R$.

We have $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n$. But note that I_k is finitely generated as an ideal of R, but since R is Noetherian, that chain stabilizes for some n.

Define a finite set of generators of I. Let $S_0 \subset R[x]$ be a finite set of polynomials that generate I_0. Define $S_1, S_2, \dots, S_n$ similarly.

Let $(S) = \bigcup_{i=0}^{n} S_i$, then take $f = a_m x^m + a_{m-1}x^{m-1} + \cdots \in I$ and $g \in (S)$.

If $m < n$, then find $a_m \in I_m$, and we are done.

If $m \geq n$, then multiply g by x^{m-n}, then kill the x^m terms (do $f - g$, where subtraction will also land to be in the ideals), until we get to the zero polynomial (just like the Euclidean algorithm). □

Then, by induction ($\implies$) follows.

§1 Short exact sequences

Definition 1.1. A module M consists of an abelian group under addition and multiplication. (Basically vector spaces over a ring.)

That is, $\forall r, s \in R$ and $\forall x, y \in M$,

- $r(x + y) = rx + ry$
- $(r + s)(x) = rx + sx$
- $(rs)x = r(sx)$
- $1 \cdot x = x$

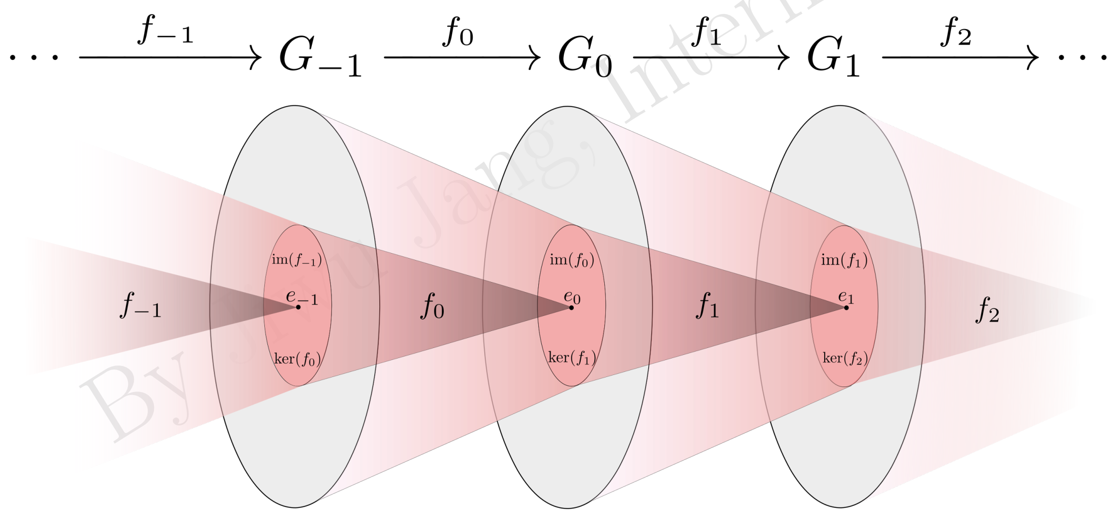

Figure 1: A depiction of an exact sequence.

Definition 1.2 (Short exact sequences (SES))**.** Given modules A, B, C, we define a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, where $\alpha : A \to B$ is injective and $\beta : B \to C$ is surjective, and $\operatorname{Ker} \beta = \operatorname{Im} \alpha$.

Definition 1.3. Define $H_n(M_0) := {}^{\operatorname{Ker} f_i}\!/\!_{\operatorname{Im} f_{i-1}}$, which is called the n^{th} homology group. The elements of H_n are called *homology classes*.

Example 1.4
Let R be a ring, and I an ideal. Then $0 \to I \to R \to R/I \to 0$ is a short exact sequence.

Example 1.5
$C \cong {}^{B}\!/\!_{\operatorname{Im}\alpha} = {}^{B}\!/\!_{\operatorname{Ker}\beta}$

Example 1.6
$0 \to I \cap J \to I \oplus J \to I + J \to 0.$

Example 1.7
$I \oplus J = (i_1, j_1) + (i_2, j_2) = (i_1 + i_2, j_1 + j_2)$ and $I + J = (1)$.

Example 1.8
$0 \to {}^{R}\!/\!_{I \cap J} \to {}^{R}\!/\!_{I} \oplus {}^{R}\!/\!_{J} \to {}^{R}\!/\!_{I + J} \to 0.$

Corollary 1.9
Chinese remainder theorem. Take $R = \mathbb{Z}$, $I = \mathbb{Z}_p$, and $J = \mathbb{Z}_q$, where $\gcd(p, q) = 1$, then CRT follows. (Super overkill.)

Proof. $0 \to {}^{\mathbb{Z}}\!/\!_{pq\mathbb{Z}} \to {}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}} \oplus {}^{\mathbb{Z}}\!/\!_{q\mathbb{Z}} \to {}^{\mathbb{Z}}\!/\!_{\mathbb{Z}} \to 0.$ □

Theorem 1.10
Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of modules. Then, M is Noetherian if and only if M' and M'' are. Moreover, M is Artinian if and only if M' and M'' are.

§2 Homology classes

Remark. The motivation comes from physics.

Theorem 2.1 (Formulation of Green's theorem)
For two paths β and β', we have $\int_\beta M dx + N dy = \int_{\beta'} M dx + N dy$ if and only if $\int_\gamma M dx + N dy = 0$ for $\gamma := \beta \oplus (-\beta')$.

Lemma 2.2

Define an equivalence class $[\beta]$ on β such that two paths are homology equivalent if and only if they have the same path integral.

Definition 2.3. For a ring R and 1 being its multiplicative identity, a *left R-module* M consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all r, s in R and x, y in M, we have

1. $r \cdot (x + y) = r \cdot x + r \cdot y$
2. $(r + s) \cdot x = r \cdot x + s \cdot x$
3. $(rs) \cdot x = r \cdot (s \cdot x)$
4. $1 \cdot x = x$.

Definition 2.4 (The universal property)**.** Let M, N be R-modules. A tensor product is an R-module P and a bilinear map $\beta : M \times N \to P$ such that $M \times N \to Q$, and we have a unique factorization through an R-module homomorphism f, where $\beta = f \circ \beta_0$.

$$\begin{tikzcd}
M \times N \arrow[r, "\beta_0"] \arrow[dr, "\beta"'] & P = M \otimes N \arrow[d, dashed, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & Q
\end{tikzcd}$$

Figure 2: A commutative diagram on a tensor product.

Example 2.5

Take a basis $e_1 = (1; 0)$ and $e_2 = (0; 1)$, then $(ae_1 + ce_2) \otimes (a'e_1 + c'e_2) = aa'e_1 \otimes e_1 + ac'e_1 \otimes e_2 + a'ce_2 \otimes e_1 + cc'e_2 \otimes e_2$. Then, $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$ form a basis.

Remark. An interesting property of tensor products is that the eigenvalues of the tensor product are the pairwise products of the eigenvalues of the individual matrices.

- $M \otimes N \cong N \otimes M$
- $(M \otimes N) \oplus P \cong (M \oplus P) \otimes (N \oplus P)$
- $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$
- $R \otimes_R M \cong M$