Banach and Hilbert Spaces

An introduction to functional analysis

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Outline

What is functional analysis?

Functional analysis is a branch of mathematical analysis that studies vector spaces endowed with a limit-related structure, such as an inner product, norm, or topology, and the linear functions defined on these spaces.

Why do we care about functional analysis?

Functional analysis is the “correct” extension of linear algebra to handle infinite dimensions (think of rank-nullity, which works only for finite-dimensional vector spaces). Infinite dimensional spaces appear naturally in analysis and topology — families of functions that appear as solutions of a differential equation, etc.

Why do we care about functional analysis?

All the classical linear PDEs can be written in the form: linear partial differential operator, acting on unknown function, yields known function. So, we can think of solving the PDE as equivalent to finding an inverse to the linear operator, just as how one would do if it were a matrix.

Why do we care about functional analysis?

But classical topological linear algebra isn't enough, since these functions could be pretty unwieldy (and they are); they might not even be continuous at all. Thus, in order to tackle these PDEs, we need a better, more refined theory of operators, that is, functional analysis. This is why Banach space theory and operator analysis emerged.

Why do we care about functional analysis?

Also, classical eigenvalue and eigenvector theory on finite-dimensional vector spaces can be generalized to infinite-dimensional spaces, which is absolutely cool. This is called spectral theory. We will later see how spectral theory generalizes eigenvalues, but first, we start with the definitions.

Normed spaces

Definition (Normed space)

Let X be a vector space over $\mathbb{C}$. Then, X is called a *normed vector space* if there exists a *norm* $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ such that $\forall f \in X$,

1. $\|f\| \geq 0$
2. $\|f\| = 0 \iff f = 0$
3. $\|cf\| = |c|\|f\|$ for all $c \in \mathbb{C}$
4. $\|f + g\| \leq \|f\| + \|g\|$

Convergent and Cauchy sequences

Definition (Convergent and Cauchy sequences)

Let X be a normed space, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.

- $\{f_n\}_{n\in\mathbb{N}}$ *converges to* $f \in X$ if

$$\forall \varepsilon > 0, \quad \exists N > 0, \quad \forall n \geq N, \quad \|f - f_n\| < \varepsilon$$

  whence $\lim_{n\to\infty} f_n = f$ or $f_n \to f$.

- $\{f_n\}_{n\in\mathbb{N}}$ is *Cauchy* if

$$\forall \varepsilon > 0, \quad \exists N > 0, \quad \forall m, n \geq N, \quad \|f_m - f_n\| < \varepsilon$$

When does Cauchy imply convergence?

It's not too hard to show that convergent sequences are Cauchy in a normed vector space. But is the converse true? In order to do analysis on infinite-dimensional vector spaces, it would be very nice to have the converse be true: all Cauchy sequences necessarily converge.

Problem

Prove that the converse direction does not hold in general.

Banach spaces

Definition (Banach space)

A normed vector space X is called *complete* if all Cauchy sequences are necessarily convergent. A complete normed vector space is called a *Banach space*.

Why do we care about Banach spaces? Because they provide a vastly generalized framework to perform analysis on families of functions, yet are still "nice enough" to be able to actually do analysis (in this matter, Hilbert spaces are better, which we will introduce later on).

Examples of Banach spaces

Let $p \in [1, \infty)$ and I be a countable index set. The following are prototypical examples of Banach spaces. The function space

$$L^p(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x < \infty \right\}$$

with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x \right)^{1/p}$$

is a Banach space.

Problem

Prove that $L^p(\mathbb{R})$ is indeed a normed vector space. (Hint: Use Minkowski's inequality)

Examples of Banach spaces

Let $p \in [1, \infty)$ and I be a countable index set. The following are prototypical examples of Banach spaces. The sequence space

$$\ell^p(I) = \left\{ (a_n)_{n \in I} : \sum_{n \in I} |a_n|^p < \infty \right\}$$

with the norm

$$\|(a_n)\|_p = \left(\sum_{n \in I} |a_n|^p \right)^{1/p}$$

is a Banach space.

Hilbert spaces

Definition (Inner product)

Let H be a vector space. Then, H is an *inner product space* if $\forall f, g \in H$, there exists an *inner product* $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ such that

a. $\langle f, f \rangle \in \mathbb{R}$ and $\langle f, f \rangle \geq 0$
b. $\langle f, f \rangle = 0 \iff f = 0$
c. $\langle g, f \rangle = \overline{\langle f, g \rangle}$
d. $\langle af_1 + bf_2, g \rangle = a \langle f_1, g \rangle + b \langle f_2, g \rangle$

Canonically, each inner product has its associated norm by the formula $\|f\| = \langle f, f \rangle^{1/2}$, and consequently, every inner product space is a normed vector space.

Hilbert spaces

Definition

If an inner product space H is complete, then it is called a *Hilbert space*. Indeed, a Hilbert space is a Banach space whose norm is determined by an inner product.

Note that the Cauchy-Schwarz inequality states that $|\langle f, g\rangle| \leq \|f\|\|g\|$ for every $f, g \in H$.

Example of Hilbert spaces

- $L^2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x$$

- $\ell^2(I)$ is a Hilbert space with the inner product

$$\langle (a_n), (b_n)\rangle = \sum_{n\in I} a_n \bar{b}_n$$

Problem

Given that $L^2(\mathbb{R})$ is a Banach space, prove that $L^2(\mathbb{R})$ is indeed a Hilbert space, that is, $\langle f, g\rangle$ is an inner product.

Spectral theory

Now, let's see how spectral theory generalizes classical eigenvalue theory.

Definition

A *bounded linear operator* is a linear transformation $T : X \to Y$ between normed vector spaces X and Y, such that $\exists M > 0$ such that $\forall x \in X$,

$$\frac{\|Tx\|_Y}{\|x\|_X} \leq M$$

where $\|T\| := \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X}$ is called the *operator norm* of T.

Spectral theory

Definition

The inverse T^{-1} of the operator T is defined by $TT^{-1} = T^{-1}T = I$. If the inverse exists, T is called *regular*. Otherwise, T is called *singular*.

Spectral theory

For a bounded linear operator T defined over a Banach space, define the operator

$$R_\zeta = (\zeta I - T)^{-1}$$

Definition

The *resolvent set* of T, denoted as $\rho(T)$, is the set of all $\zeta \in \mathbb{C}$ such that R_ζ exists and is bounded.

Definition

The *spectrum* of T, denoted as $\sigma(T)$, is the set of all $\zeta \in \mathbb{C}$ such that R_ζ fails to exist or is unbounded. (This is analogous to the finite-dimensional case when $p(\lambda) = \det(A - \lambda I)$ fails to have an inverse for its eigenvalues.)

Spectral theory

Indeed, the spectrum of T is the complement of the resolvent set of T in $\mathbb{C}$. Every eigenvalue of T belongs to $\sigma(T)$, but $\sigma(T)$ may contain non-eigenvalues (namely, those that give unbounded operators).

For further reference, one may consult the following:

- *Linear Operators, Part 1: General Theory* by Dunford and Schwartz.
- *Linear Operators, Part 2: Spectral Theory* by Dunford and Schwartz.
- *Functional analysis*, Riesz and Szőkefalvi-Nagy

Thank you!