The multifarious Cantor set

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This is a note on a series of lectures on Cantor sets, given by Prof. Vitaly Bergelson, accompanied by various "What is . . . ?" seminars at Ohio State University.

§1 Cantor set

Definition 1.1. The Cantor set $\mathcal{C}$ is created by iteratively deleting the open middle third from a set of line segments.

One starts by deleting the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ from the interval $[0, 1]$, leaving two line segments: $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

The Cantor set contains all points in the interval $[0, 1]$ that are not deleted at any step in this infinite process.

Formally, we define $\mathcal{C} := \bigcap_{i=1}^{\infty} \mathcal{C}_i$ where $\mathcal{C}_i$ is the set after each iteration.

Definition 1.2. We define $\{0, 1, 2, \ldots, r-1\}^{\mathbb{N}}$ to be the set of all infinite sequences $(x_n)_{n\in\mathbb{N}}$ with entries from $\{0, 1, 2, \ldots, r-1\}$.

Definition 1.3. Alternatively, we define

$$\mathcal{C} := \left\{ \sum_{i=1}^{\infty} \frac{t_i}{3^i} \mid t_i \in \{0, 2\} \right\}$$

that is, we take all ternary expansions of the number $x \in [0, 1]$ whose digits only consist of $\{0, 2\}$. Hence, by Cantor's diagonalization argument, $\mathcal{C}$ is uncountable.

In other words, $|\mathcal{C}| = |\{0, 2\}^{\mathbb{N}}| = |2^{\mathbb{N}}| = |\mathbb{R}|$.

Moreover, there is a natural map between $\mathcal{C} \cong \{0, 2\}^{\mathbb{N}}$, where $x \in \mathcal{C}$ if and only if there exists a ternary expansion of x that only uses the digits 0 and 2, and if there are multiple ternary expansions, then at most one can only use the digits 0 and 2.

Definition 1.4. We say a set S is *countable* if a bijection can be formed between the sets S and $\mathbb{N}$.

> **Theorem 1.5**
> $\mathbb{Z} \times \mathbb{Z}$ is countable. (Thus, $\mathbb{Q}$ is countable as well.)

Proof. Consider the spiral walk starting at $(0, 0)$, visiting every element in the lattice plane. Thus, we have formed a bijection. □

Theorem 1.6 (Cantor's diagonalization method)
$\mathcal{C}$ is uncountable.

Proof. Assume for the sake of contradiction that $\mathcal{C}$ was countable.
Then, we may write down the elements in $\mathcal{C}$ as follows:

$$\mathcal{C} = \begin{cases} a_{11}a_{12}a_{13}\dots \\ a_{21}a_{22}a_{23}\dots \\ a_{31}a_{32}a_{33}\dots \\ \vdots \end{cases}$$

Consider $\tilde{a}_{11}, \tilde{a}_{22}, \dots$ where each $\tilde{a}_{ii}$ is the "flip" of a_{ii}. Then, the flipped sequence cannot appear anywhere in our table, because it must meet with the diagonal, yet when they meet the digits differ, contradiction. □

§1.1 Exercises and Problems

Definition 1.7. For a binary operator $*$, define $\mathcal{C} * \mathcal{C} := \{x * y \mid x, y \in \mathcal{C}\}$ where $x * y$ is properly defined. (e.g. no division by zero.)

Exercise 1.8. Show $\mathcal{C} + \mathcal{C} = [0, 2]$ and $\mathcal{C} - \mathcal{C} = [-1, 1]$.

Proof. For $\mathcal{C} + \mathcal{C}$, let $S = \{x + y \mid x, y \in \mathcal{C}/2\}$, and note that

$$S = \{x + y \mid x = \sum_{i=1}^{\infty} \frac{a_i}{3^i};\ y = \sum_{i=1}^{\infty} \frac{b_i}{3^i} \text{ where } a_i, b_i \in \{0, 1\}\}$$

but for each digit of $x+y$, we have $\{0, 1\} + \{0, 1\} = \{0, 1, 2\}$, thus we can represent every number in $[0, 1]$, since every number in $[0, 1]$ has a base-3 expansion $0.c_1c_2\dots_{(3)}$ where $c_i \in \{0, 1, 2\}$. Hence, $S = [0, 1]$. Now, simply multiplying each element in S by two gives $\mathcal{C} + \mathcal{C} = 2S = [0, 2]$, and we are done.

For $\mathcal{C} - \mathcal{C}$, the statement is very similar, but we shall perform the balanced ternary expansion. Let $S = \{x - y \mid x,\ y \in \mathcal{C}/2\}$, and now note that

$$S = \{x - y \mid x = \sum_{i=1}^{\infty} \frac{a_i}{3^i};\ y = \sum_{i=1}^{\infty} \frac{b_i}{3^i} \text{ where } a_i, b_i \in \{0, 1\}\}$$

but for each digit of $x + y$, we have $\{0, 1\} - \{0, 1\} = \{-1, 0, 1\}$, thus we can represent every number in $[-\frac{1}{2}, \frac{1}{2}]$, since every number in $[-\frac{1}{2}, \frac{1}{2}]$ has a balanced base-3 expansion $0.c_1c_2\dots_{(3)}$ where $c_i \in \{-1, 0, 1\}$. Hence, $S = [-\frac{1}{2}, \frac{1}{2}]$. Now, simply multiplying each element in S by two gives $\mathcal{C} - \mathcal{C} = 2S = [-1, 1]$, and we are done. □

Problem 1.9. Show $\mathcal{C} \cdot \mathcal{C} = [0, 1]$ and $\mathcal{C}/\mathcal{C} = \mathbb{R}^{\geq 0}$.

Proof. For $\mathcal{C} \cdot \mathcal{C}$, we have $\{0, 2\} \cdot \{0, 2\} = \{0, 2, 4\}$, but in modulo 3 it is equivalent to $\{0, 1, 2\}$ except for the fact that it adds a carry; but that can be handled with induction until the k^{th} digit, so $\mathcal{C} \cdot \mathcal{C} = [0, 1]$ since it includes every base-3 expansion of elements in $[0, 1]$.

For $\mathcal{C}/\mathcal{C}$, intuitively, although $\mathcal{C}$ does not produce every real number in $[0, 1]$, for a given positive real number r, we may create a sequence that approximates and converges to r, by "enhancing the approximation" in every step of adding another digit. (We should formalize this notion.) □

Exercise 1.10. Show that $\frac{1}{4} \in \mathcal{C}$.

Proof. We have $\frac{1}{4} = 2\left(\sum_{k=1}^{\infty} \frac{1}{9^k}\right)$, thus we are done. □

Exercise 1.11. Show that $\frac{1}{\sqrt{2}} \notin \mathcal{C}$.

Problem 1.12. Is there a quadratic irrational in $\mathcal{C}$?

Exercise 1.13. Prove that $|[0,1]| = |\mathbb{R}|$. We say that $[0,1]$ is *equinumerous* with $\mathbb{R}$.

Proof. Use $\tan^{-1}((x - \frac{1}{2})\pi)$ to create a bijection between $(0,1)$ and $\mathbb{R}$, then since $|[0,1]| = |(0,1)|$ because $(0,1)$ only excludes two points 0 and 1, conclude. □

Exercise 1.14. Prove that $|[0,1] \times [0,1]| = |[0,1]|$.

Proof. Consider two arbitrary numbers in $[0,1]$, let them be a and b. Then, take the binary expansion of $a = 0.a_1a_2a_3\ldots$ and $b = 0.b_1b_2b_3\ldots$. Moreover, send it to $c = 0.a_1b_1a_2b_2a_3b_3\ldots$, and take the canonical representation only. Thus we formed a bijection, and we are done. □

Exercise 1.15. Prove that $|[0,1]| = |\mathbb{R} \times \mathbb{R}|$.

Proof. By the previous two exercises, we are done. □

§2 Properties of $\mathcal{C}$

Here are some properties of $\mathcal{C}$:

1. $\mathcal{C}$ is closed. (Intersection of arbitrarily many closed sets is closed.)
2. $\mathcal{C}$ is *nowhere dense*. In particular, $\mathcal{C}$ does not contain $[a,b]$ for some $0 \le a < b \le 1$.
3. $\mathcal{C}$ is "arithmetically large".
4. $\mathcal{C}$ has measure zero ($\int_{\mathcal{C}} d\mu = 0$), since its complement has measure one.
5. $\mathcal{C}$ is uncountable. ($|C| = |R| = |[0,1]|$)

> **Remark.** $\mathcal{C}$ is *large* in some senses, but is also *small* in other senses.

Definition 2.1. A *Borel set* is any set in a topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of

- countable union
- countable intersection
- relative complement (with respect to the parent set)

Definition 2.2. For a topological space X, the collection of all Borel sets on X forms a σ-algebra, known as the *Borel σ-algebra*, often denoted as $\mathfrak{B}$. The Borel σ-algebra on X is the smallest σ-algebra containing all open sets (or, equivalently, all closed sets).

Theorem 2.3

We have $\mathcal{C} \in \mathfrak{B}$, where $\mathfrak{B}$ is the Borel σ-algebra.

Proof. $\mathcal{C}$ is a countable intersection of closed sets $\mathcal{C}_i$, and thus $\mathcal{C}$ is closed. Hence $\mathcal{C} \in \mathfrak{B}$ by definition of a Borel set. □

Theorem 2.4

The set of algebraic numbers is countable, hence there exists transcendental numbers in $\mathbb{R}$ and beyond. (e.g. π and e are transcendental.)

Proof. This follows from some observations:

1. The union of countably many countable sets is countable. (e.g. $\mathbb{Z} \times \mathbb{Z}$)
2. Fix degree $d \geq 0$, and consider S_d, which are the set of all possible roots of degree d polynomials with integer coefficients.
3. The polynomial $a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ for $a_d \neq 0$ has at most d roots (Fundamental Theorem of Algebra).

Now, since each coefficient is countable, the union of countably many such set of coefficients is countable, hence S_d is countable as well. □

§3 Devil's staircase function

Definition 3.1. The Cantor function, also known as the "devil's staircase function", is defined on the complement of Cantor set in $[0, 1]$, then extended by "filling in" the removed intervals by continuity.

Formally, the Cantor function $c : [0, 1] \to [0, 1]$ is defined as

$$c(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{2^n}, & x = \displaystyle\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in \mathcal{C} \text{ for } a_n \in \{0, 1\}; \\ \displaystyle\sup_{y \leq x,\, y \in \mathcal{C}} c(y), & x \in [0, 1] \setminus \mathcal{C}. \end{cases}$$

Definition 3.2. A set $S \subseteq \mathbb{R}$ has measure zero if $\forall \varepsilon > 0$, it can be covered by a finite or countable family of intervals $\mathcal{I}$ with total length $< \varepsilon$.

Formally, $\forall \varepsilon > 0$, $\exists \mathcal{I} = \{I_1, I_2, \dots\}$ where $I_i \subset \mathbb{R}$ s.t. $S \subseteq \bigcup_{I \in \mathcal{I}} I$ and $\sum_{I \in \mathcal{I}} |I| < \varepsilon$ where $|I|$ denotes the length (and measure) of interval I.

Example 3.3

$\mathbb{Q}$ has measure zero.

Proof. Let $\mathbb{Q} = \{r_1, r_2, \dots\}$ since $\mathbb{Q}$ is countable.

Then, for some intervals J_i such that $r_i \in J_i$, we have $|J_i| < \frac{\varepsilon}{2^{i+1}}$.

Thus, $\sum |J_i| < \varepsilon$ for all $\varepsilon > 0$.

Hence, $\mathbb{Q}$ has measure zero. □

Example 3.4

$\mathcal{C}$ has measure zero (which we previously handwaved this by saying that its complement is of length 1).

Proof. Because $\mathcal{C} \subseteq \mathcal{C}_i$, we may consider the measure of the interval $\mathcal{C}_i$: observe that the total length of $\mathcal{C}_i$ is bounded above by $\varepsilon = \frac{1}{3^i}$, hence $\mu(\mathcal{C}) = 0$. □

Here are some properties that the Cantor function $c(x)$ satisfies:

1. $c'(x) = 0$ almost everywhere, meaning that $\{x \mid c'(x) = 0\}$ is of measure zero.

§3.1 Exercises and Problems

Exercise 3.5. Prove that the arc length of $c(x)$ is 2.

Proof. Since $c(x)$ is continuous on $[0, 1]$, by using the triangle inequality, we get that the arc length of $c(x)$ is at most $1 + 1 = 2$.

Now, the union of the segments for which $c'(x) = 0$ is of measure 1, and for every other point for which the derivative is not defined, i.e., $x \in \mathcal{C}$, we project it to the y-axis. Then, since each projected number on the y-axis is $\sum_{i=1}^{\infty} \frac{a_i}{2^i}$ where $a_i \in \{-1, 1\}$, it is equivalent to $[0, 1]$. Hence, the arc length is at least 2.

Thus, the arc length of $c(x)$ is 2. □

Exercise 3.6. Show that it does not matter if we choose J_τ to be open, closed, half-open, or half-closed.

Proof. Changing a side of an interval from closed to open excludes only one point, which has measure zero, so the result follows. □

Exercise 3.7. Define (carefully) the notion of $\mu = 0$ in $\mathbb{R}^n$ and show that all reasonable definitions are equivalent.

Definition 3.8. We say that a set $S \subseteq \mathbb{R}^n$ has measure zero, denoted as $\mu(S) = 0$, if $\forall \varepsilon > 0$, $\exists \mathcal{B} = \{B_1, B_2, \dots\}$ where $B_i \subseteq \mathbb{R}^n$ s.t. $S \subseteq \bigcup_{B \in \mathcal{B}} B$ and $\sum_{B \in \mathcal{B}} |B| < \varepsilon$ where $|B|$ denotes the measure of the open ball B.

Exercise 3.9. $c'(x)$ is zero on a measure 1 subset of $[0, 1]$. In particular, it is 0 on the interior of the complement of $\mathcal{C}$.

Proof. Because $\mu(A) = \mu(A - B) + \mu(B)$ for any measurable sets A and B, we have

$$\mu([0, 1]) = \mu([0, 1] - \mathcal{C}) + \mu(\mathcal{C})$$

hence $1 = \mu([0, 1] - \mathcal{C}) + 0$, but $c'(x) = 0$ for all $x \in [0, 1] - \mathcal{C}$, thus we are done. □

Exercise 3.10. Prove that $\displaystyle\int_{[0,1]} c'(x)\, d\mu = 0$.

Proof. The derivative of the Cantor function, denoted as $c'(x)$, exists almost everywhere and is zero almost everywhere, except for the Cantor set, where it is undefined.

To find the integral of $c'(x)$ over $[0, 1]$, we need to consider the Lebesgue integral. The Lebesgue integral takes into account the measure of sets when integrating functions.

Since $c'(x)$ is zero almost everywhere, we can consider the integral over the complement of the Cantor set. Let's denote the complement of the Cantor set as A.

Then, $\mu(A) = \mu([0,1]) - \mu(\mathcal{C}) = 1$ since A is the entire interval $[0,1] \setminus \mathcal{C}$ and $\mu(\mathcal{C}) = 0$. Now, integrate $c'(x)$ over A, that is,

$$\int_A c'(x)\, d\mu = \int_A 0\, d\mu = 0$$

hence,

$$\int_{[0,1]} c'(x)\, d\mu = \int_A c'(x)\, d\mu = 0$$

and we are done. □

> **Remark.** The function f' is certainly *not* Riemann-integrable, since it is undefined at the Cantor set.

Exercise 3.11.

$$\mathbb{1}_{\mathcal{C}} = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{if } x \notin \mathcal{C} \end{cases}$$

is Riemann-integrable.

Exercise 3.12. For any countable set $S \subseteq \mathbb{R}$, there exists a monotone function $f : \mathbb{R} \to \mathbb{R}$ such that the set of its discontinuities is S.

Exercise 3.13. Most continuous functions are nowhere differentiable.

> **Remark.** Monotone functions are much better than general continuous functions.

Exercise 3.14. Any monotone function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.

§4 Metric spaces

Definition 4.1. A set $X \neq \emptyset$ is called a metric space, if there is a metric $d : X \times X \to [0, \infty)$ such that

1. $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x) \quad \forall x, y \in X$
3. $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x, y, z \in X$ (triangle inequality)

Example 4.2

Here are some examples of metric spaces:

- Any set X equipped with the discrete metric

$$d(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

- $X = C[0,1]$ equipped with the metric

$$d(f,g) = \max_x (f(x) - g(x))$$

  ($C[0,1]$ denotes the set of all continuous functions $f : [0,1] \to \mathbb{R}$)

- $X = \mathbb{R}^n$, equipped with the metric

$$d_p(x,y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}$$

  for sequences $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ where $p \in [1, \infty)$. (note that we do not include $p = \infty$, which we define below.)

- $X = \mathbb{R}^n$, equipped with the metric $d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|$.

- $\ell_2 = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{R}\ \forall i \geq 1 \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ equipped with the metric

$$d(x,y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

- $X = \{0,1\}^{\mathbb{N}}$, equipped with the metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

  where $x, y \in \{0,1\}^{\mathbb{N}}$. (the p-adic metric.)

§4.1 Exercises and Problems

Exercise 4.3. Prove that $\lim_{p\to\infty} d_p = d_\infty$.

Proof. Let $a_i = |x_i - y_i|$ and $a = \max_{i=1}^n a_i$, then $d_p > a$ and $d_p < (a^p n)^{\frac{1}{p}} = a n^{\frac{1}{p}}$, and thus $\lim_{p\to\infty} d_p = a$ by the squeeze theorem, so $\lim_{p\to\infty} d_p = d_\infty$. □

Exercise 4.4. Prove that d_p is indeed a metric on $X = \mathbb{R}^n$.

Proof. Use Minkowski's inequality to prove that d_p satisfies the triangle inequality; other properties are easy to show. (Fun exercise, try deriving Minkowski's inequality from Hölder's inequality.) □

Exercise 4.5. Is there $\varepsilon > 0$ such that $\mathcal{C} \cap (\mathcal{C} - \varepsilon) \neq \emptyset$ for $\varepsilon < \frac{1}{2}$?

Proof. We may shift everything by $\varepsilon = \frac{2}{3^2}$, which would obviously lend some numbers to be in the intersection, hence we are done. □

Theorem 4.6

A function $f : [0,1] \to \mathbb{R}$ is Riemann-integrable if and only if it is bounded and the set of discontinuities of f has measure zero.

Exercise 4.7. Prove that any monotone function has at most countably many discontinuities.

Exercise 4.8. Prove that any continuous function is Riemann-integrable.

Exercise 4.9. Show that any monotone function is Riemann-integrable.

Exercise 4.10. Prove that there exists a *strictly monotone* function $f : [0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$, and $f'(x) = 0$ for almost every $x \in [0,1]$.

Proof. Let φ be a generalized Cantor step function. Take

$$f(x) = c \cdot \sum_{i=1}^{\infty} \frac{\varphi(nx)}{2^n}$$

where $f(0) = 0$ and

$$f(1) = c \cdot \sum_{i=1}^{\infty} \frac{\varphi(n)}{2^n}$$

where c is such that $f(1) = 1$. Then,

$$f'(x) = \left(c \cdot \sum_{n=1}^{\infty} \frac{\varphi(nx)}{2^n} \right)' = c \cdot \sum \frac{n \cdot \varphi'(nx)}{2^n}$$

by little Fubini's theorem. For $x_1 < x_2$, pick n such that nx_1 and nx_2 lie on different unit intervals, then $\varphi(nx_1 + 1) \leq \varphi(nx_2)$, so we are done. □

Problem 4.11. Is there a continuous yet nowhere monotone function?

> **Remark** (Musings)**.** An example of a continuous yet nowhere differentiable function is the Weierstrass functions $f := \sum a^n \sin(b^n x)$.

Definition 4.12. Two metric spaces (X_1, d_1) and (X_2, d_2) are *isometric* if $\exists \varphi : X_1 \to X_2$ such that $\forall a, b \in X_1$,

$$d_2(\varphi(a), \varphi(b)) = d_1(a, b)$$

> **Remark.** Isometry is the simplest form of isomorphism. (There are other types of isomorphisms as well.)

Definition 4.13. We say two sets X_1 and X_2 are *homeomorphic* if $\exists \varphi : X_1 \to X_2$ s.t. φ is bijective and bicontinuous (i.e., φ is continuous, and its inverse is also continuous).

Corollary 4.14

Homeomorphisms preserve limits, that is,

$$\varphi(\lim a_n) = \lim(\varphi(a_n))$$
$$\varphi^{-1}(\lim b_n) = \lim(\varphi^{-1}(b_n))$$

where $(a_n) \subseteq X_1$ and $(b_n) \subseteq X_2$.

Example 4.15

$[0, 1]$ and $\mathbb{R}$ are homeomorphic, since we may consider the ray from the half circle to the real line, where maps are bijective and bicontinuous.

Example 4.16

$(0, 1)$ and $[0, 1]$ are not homeomorphic, since $[0, 1]$ is compact while $(0, 1)$ is not.

Example 4.17

$(\{0, 1\}^{\mathbb{N}}, d) \sim \mathcal{C}$, since $\exists \varphi : \mathcal{C} \to \{0, 1\}^{\mathbb{N}}$ such that $0 \mapsto 0$ and $2 \mapsto 1$, which preserves the distance metric between $\mathcal{C} \sim \{0, 2\}^{\mathbb{N}} \sim \{0, 1\}^{\mathbb{N}}$ (hence they are homeomorphic).

Example 4.18

Let (X, d) be a metric. Then X can be given a topology via $B_r(x) := \{y \mid d(x, y) < r\}$ is open $\forall r \in \mathbb{R}^+$ and $x \in X$.

Definition 4.19. A topological space is Hausdorff (T2-separable) if $\forall x, y\ \exists U_1 \ni x, U_2 \ni y$ such that $U_1 \cap U_2 = \emptyset$.

Exercise 4.20. For a compact Hausdorff topology, one cannot remove or add points and preserve both compactness and Hausdorff.

Exercise 4.21. Prove that if X, Y are Hausdorff, then $X \times Y$ is also Hausdorff.

Exercise 4.22. Find a topology τ of $\mathbb{R}$ such that $\forall f : \mathbb{R} \to \mathbb{R}$, f is continuous under $\mathbb{R}_\tau \to \mathbb{R}_{\text{standard}}$.

Take $\tau = 2^{\mathbb{R}}$.

Exercise 4.23. Find a topology τ of $\mathbb{R}$ such that $\forall f : \mathbb{R} \to \mathbb{R}$, f is continuous under $\mathbb{R}_{\text{standard}} \to \mathbb{R}_\tau$.

Take $\tau = \{\emptyset, X\}$ (the trivial topology).

Exercise 4.24. Prove that if $\{A_i\}_{i=1}^{\infty}$ is a family of $|\mathbb{R}|$ sets, then $|\bigcup A_i| = |\mathbb{R}|$.

Exercise 4.25. Let $\pi_1 : X \times Y \to X$ via $(x, y) \mapsto x$. Show that π_1 is continuous. Define π_2.

Proof. Let $U \subseteq X$ be open. Then, $\pi^{-1}(U) = \{(x, y) \mid x \in U\} = U \times Y$, which is open. □

Theorem 4.26

Let (X, τ) be a topology of a metric space. Then, the following two definitions of compactness are equivalent:

- Let $A \subseteq X$ such that $|A| = \infty$. Then A has a limit point.
- Let $\{x_i\}_{i=1}^{\infty} \subseteq X$, then $\exists i_k$ where $k \in \mathbb{N}$ such that x_{i_k} converges to $x \in X$.

Definition 4.27. Consider the function $f(x) = \lambda x(1-x)$ where $\lambda > 4$. We delete the interval of x such that $f(x) > 1$. Repeat this iteration infinitely many times, and observe the part that still remains. We call this the *generalized Cantor set*, denoted as $\mathcal{C}_\lambda$.

Example 4.28

The classical middle 3^{rd} Cantor set $\mathcal{C}$ is actually $\mathcal{C}_{\frac{9}{2}}$.

Exercise 4.29. Prove that the generalized Cantor set $\mathcal{C}_\lambda$ has measure zero, $\forall \lambda > 4$.

Proof. Denote the measure of the deleted open intervals in the n^{th} iteration as a_n. Then, for the first iteration, we delete the interval

$$\left(\frac{1 - \sqrt{1 - \frac{4}{\lambda}}}{2}, \frac{1 + \sqrt{1 - \frac{4}{\lambda}}}{2} \right)$$

which is of measure $\sqrt{1 - \frac{4}{\lambda}}$, which we denote as k. Then, since k is also the proportion of the measure of the deleted interval over the entire interval $[0, 1]$, we have $a_1 = k$, $a_2 = 2^1 \cdot \frac{1-k}{2} \cdot k$, and in general,

$$a_n = 2^{n-1} k \left(\frac{1-k}{2} \right)^{n-1} = k(1-k)^{n-1}$$

But then, $\sum_{n=1}^{\infty} a_n = 1$ regardless of k, hence the complement of the deleted intervals, which is precisely $\mathcal{C}_\lambda$, has measure zero. □

Exercise 4.30. Prove that all such generalized Cantor sets are homeomorphic to the classical middle 3^{rd} Cantor set.

Proof. The n^{th} iteration generates exactly 2^n boundary points, all of which are totally disconnected; hence we may form a bicontinuous bijection between each point in the Cantor set $\mathcal{C}$ and the generalized Cantor set $\mathcal{C}_\lambda$. □

§5 Perfect sets

Definition 5.1. A point x is called an *isolated point* of a subset S (in a topological space X) if x is an element of S and there exists a neighborhood of x that does not contain any other points of S.

This is equivalent to saying that the singleton $\{x\}$ is an open set in the topological space S (considered as a subspace of X).

Another equivalent formulation is the following: an element x of S is an isolated point of S if and only if it is not a limit point of S.

Definition 5.2. A subset of a topological space (X, τ) is said to be *perfect* if it is closed and has no isolated points.

Definition 5.3. A *totally disconnected* space is a topological space that has only singletons as connected subsets.

In every topological space, the singletons (and, when it is considered connected, the empty set) are connected; in a totally disconnected space, these are the only connected subsets.

Example 5.4

The Cantor set $\mathcal{C}$ is a perfect set, but is also *totally disconnected.*

Example 5.5

Other examples of perfect subsets of the $\mathbb{R}$ are the empty set, all closed intervals, and $\mathbb{R}$ itself.

§6 Szemerédi's theorem

Definition 6.1. Let $E \subseteq \mathbb{Z}$. Define the upper density of E as

$$\overline{d}(E) := \limsup_{N\to\infty} \frac{|E \cap \{-N, \dots, N\}|}{2N+1}$$

and the lower density of E as

$$\underline{d}(E) := \liminf_{N\to\infty} \frac{|E \cap \{-N, \dots, N\}|}{2N+1}$$

Definition 6.2. When $\overline{d}(E) = \underline{d}(E) = e$, $d(E) = e$ is the natural density of E.

Example 6.3

For $E = \mathbb{Z}$, $d(E) = 1$.

Example 6.4

For $E = n\mathbb{Z}$, $d(E) = \frac{1}{n}$.

Example 6.5

For $E = \bigcup_{n\geq 1} \{j \mid 2^{2n} < j < 2^{2n+1}\}$, $\overline{d}(E) = \frac{2}{3}$ and $\underline{d}(E) = \frac{1}{3}$, so $d(E)$ is undefined.

Example 6.6

$\overline{d}$ and $\underline{d}$ are not additive. (Think of $\mathbb{N} - E$ and E in the previous example. They clearly don't add up.)

Remark. However, $\overline{d}$ is sub-additive. Moreover, d is also invariant under translation.

Conjecture 6.7 (Erdös, Turán, 1936)**.** If $d(E) > 0$, then E contains arithmetic progressions of arbitrary length, i.e., $\forall k \in \mathbb{N}, \exists a \in E,\ b \in \mathbb{N}$ where $\{a + mb \mid 0 \leq m \leq k\} \subseteq E$.

Theorem 6.8 (Szemerédi, 1975)
If $\overline{d}(E) > 0$, then E contains arithmetic progressions of arbitrary length.

Example 6.9 (Szemerédi as a *shift*)
$A \cap A - k \cap A - 2k \neq \emptyset \implies A$ contains $\{x, x+k, x+2k\}$.

Definition 6.10. We define the indicator function

$$\mathbb{1}_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}$$

This is linked to Cantor sets in the following way:

Remark. For any $S \subseteq \mathbb{N}$, $\mathbb{1}_S(n) \in \{0,1\}^{\mathbb{N}} \sim \mathcal{C}$.

Theorem 6.11 (Furstenberg, 1977)
For $E \subseteq \mathbb{Z}$, then $\exists(X, \mathfrak{B}, \mu, T)$, where $(X, \mathfrak{B}, \mu)$ is a probability space and T is a measure-preserving transformation, i.e., $\mu(T^{-1}E) = \mu(E) \quad \forall E \in \mathfrak{B}$. Then, $\exists A \subseteq \mathfrak{B}$ such that $\mu(A) = \overline{d}(E)$ and

$$\overline{d}(E - n_1 \cap E - n_2 \cap \cdots \cap E - n_k) \geq \mu(T^{-n_1}A \cap T^{-n_2}A \cap \cdots \cap T^{-n_k}A) \quad \forall n_i \in \mathbb{Z}$$

Theorem 6.12 (Riesz representation)
Any positive linear functional on $C(x)$ where x is compact Hausdorff can be represented by integration with respect to some positive measure.

Theorem 6.13 (Gelfand representation)
If C is a commutative C^*-algebra with spectrum X, then $\exists \gamma : C \to C(x)$ which is an isometric $*$-isomorphism.

Theorem 6.14 (Hahn-Banach)
If $p : X \to \mathbb{R}$ is sublinear (where X is a vector space) and $f : Y \to \mathbb{R}$ and $X \supseteq Y$ with $f \leq p$, then $\exists F : X \to \mathbb{R}$ s.t. $F|_Y = f$, $F \leq P$ and F linear.

Proof. $\{E\}$ is countable, so $\{E - n \mid n \in \mathbb{Z}\}$ is countable.

Hence, $\Xi = \{E - n_1 \cap E - n_2 \cap \cdots \cap E - n_k \mid n_i \in \mathbb{Z},\ k \in \mathbb{N}\}$ is countable.
Take $E' \in \Xi$, and define

$$L(\mathbb{1}_{E'}) = \lim_{i \to \infty} \frac{|E' \cap \{-N_i, \ldots, N_i\}}{2N_i + 1}$$

We have $L(\mathbb{1}_{E'}) \leq \overline{d}(E')$, because we have a limit of a subsequence of the lim sup of the original sequence.
Note that L is additive in $\mathbb{1}_{\Xi}$ in the sense of measure, that is,

$$L(E' \cup E'') = L(E') + L(E'') \text{ for } E' \cap E = \emptyset$$

Generate an algebra $\mathcal{A}$ generated by $\mathbb{1}_{\Xi}$, and by Hahn-Banach, L extends to $\mathcal{A}$.
By Gelfand, $\mathcal{A} \cong C(x)$.
By Riesz, $L = \int - \, d\mu$.
From $\mu(A) = \overline{d}(E)$, we have $\mu(A) = \int \mathbb{1}_A d\mu$ and $\overline{d}(E) = L(E)$, hence $\int \mathbb{1}_A d\mu = L(E)$, which is true by diagonalization procedure.
Therefore,

$$\begin{aligned}
\overline{d}\left(\bigcap_{i=1}^{k} E - n_i\right) &\geq L\left(\mathbb{1}_{\bigcap i=1^k E - n_i}\right) \\
&= \int \prod \mathbb{1}_{T^{-n_i} A} d\mu \\
&= \mu(T^{-n_1} A \cap T^{-n_2} A \cap \cdots \cap T^{-n_k} A)
\end{aligned}$$

□

Corollary 6.15

Szemerédi's theorem reduces to proving for any $(X, \mathfrak{B}, \mu, T)$, $A \in \mathfrak{B}$, $\mu(A) > 0$, and $\forall k \in \mathbb{N}$, $\exists m \in \mathbb{N}$ s.t.

$$\mu(A \cap T^{-m} A \cap T^{-2m} A \cap \cdots \cap T^{-km} A) > 0$$

Proof. H. Furstenberg, "Ergodic behavior of diagonal measures and a theorem of Szemerédi." □

A generalization of ergodic multidimensional Szemerédi is as follows:

Theorem 6.16 (Ergodic multidimensional Szemerédi)

For any $(X, \mathfrak{B}, \mu, T_1, T_2, \ldots, T_k)$ where $A \in \mathfrak{B}$, $\mu(A) > 0$, and T_i commuting, there exists $m \in \mathbb{N}$ such that

$$\mu(A \cap T_1^{-m} A \cap T_2^{-m} A \cap \cdots \cap T_k^{-m} A) > 0$$

From now on, consider $A \subseteq \mathbb{Z}$, which does not change anything.
Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ and $\sigma : x(n) \to x(n+1)$.

Exercise 6.17. Prove that the underlying metric

$$d(\mathbf{x}, \mathbf{y}) := \sum_{i \in \mathbb{Z}} \frac{|x(i) - y(i)|}{2^{|i|}}$$

is indeed a metric. (Intuitively, we compare a neighborhood centered around 0 between two sequences.)

Exercise 6.18. Prove that σ is a homeomorphism of Ω.

Exercise 6.19. Prove that $\{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{Z}}$ are homeomorphic.

Example 6.20 (Iterations of σ)

Consider the set $X_A = \overline{\{\sigma^k(\mathbb{1}_A(n)) \mid k \in \mathbb{Z}\}} \subseteq \Omega = \{0,1\}^{\mathbb{Z}}$. (It's an orbital closure, which contains all of its limit points.)

Example 6.21

We have exactly two sets $A = \mathbb{Z}$ and $A = \emptyset$ such that X_A is a singleton. We have $A = 2\mathbb{Z}$ such that X_A has two elements.

Exercise 6.22 (Bruce)**.** Prove that X_A is finite if and only if A is periodic.

Exercise 6.23. Classify when X_A can be countable?

Definition 6.24. A word is a finitely many consecutive $\{0,1\}$-digits.

Exercise 6.25. Prove that $X_A = \Omega$ if and only if A (as a binary sequence) has any finite binary word as a substring.

Definition 6.26. A word ω has *correct frequency* if the frequency of ω is $\frac{1}{2^{|\omega|}}$.

Definition 6.27. $\omega \in \Omega$ is *normal* if every subword has correct frequency. For convenience, $\Omega = \{0,1\}^{\mathbb{N}}$.

Problem 6.28 (Champernowne)**.** Prove that the constant $c = 0.123456789101112\ldots$ is decimal normal.

Theorem 6.29

For any integer polynomial $f : \mathbb{Z} \to \mathbb{N}$, the sequence

$$0.f(1)f(2)f(3)\ldots$$

(where we concatenate digits) is normal.

Problem 6.30. Are most random sequences normal? What is the proportion of normal random sequences?

Problem 6.31. Is $0.1491625364964\ldots$ normal?

Problem 6.32. Is $0.235711131719232931\ldots$ normal?

Problem 6.33. Is $e = 2.71828\ldots$ normal?

Problem 6.34. Is $\pi = 3.141592\ldots$ normal?

Problem 6.35 (Euler-Mascheroni)**.** Is $\gamma = 0.577215664901532\ldots$ normal?

Theorem 6.36 (Borel)
The set of $x \in [0,1]$ whose binary expansions are normal has complement of measure zero. (i.e., almost every $x \in [0,1]$ is normal in base 2.)

Exercise 6.37. The set of normal base 2 numbers in $[0,1]$ form a set of Baire category I.

Exercise 6.38. Is a base b normal sequence also normal in base b', for $b' \neq b$?

Exercise 6.39. Prove that $\mathcal{C}$ is Borel-measurable.

Definition 6.40. $\mathfrak{B}$, by definition, is a σ-algebra of subsets of $\mathbb{R}$, which is generated by open sets. Equivalently, $\mathfrak{B}$ is generated by intervals (a,b) where $a,b \in \mathbb{Q}$.

Proof. $(\alpha,\beta) = \bigcup_{n=1}^{\infty}(a_n,b_n)$ where $a_n \to \alpha$ from the right and $b_n \to \beta$ from the left, where $a_n, b_n \in \mathbb{Q}$. □

Proof. By definition, $\mathcal{C}$ is the complement of a certain open set in $[0,1]$, so by definition $\mathcal{C} \in \mathfrak{B}$. □

Exercise 6.41. Prove that any open set in $\mathbb{R}$ is a disjoint union of open intervals.

Exercise 6.42. Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Exercise 6.43. What is $|\mathfrak{B}|$?

$\mathfrak{B}$ is countably generated by real intervals (which is countably generated by rational intervals), so a typical subset of $\mathbb{R}$ is not in $\mathfrak{B}$.

Exercise 6.44. Prove that not every subset of $\mathcal{C}$ is Borel.
That is, $\mathfrak{B} \subsetneq$ Lebesgue measurable sets.

Problem 6.45. Are there non Borel sets?

Problem 6.46. Give an example of a sequence $f_n : [0,1] \to \mathbb{R}$ of continuous functions such that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0,1]$. How badly discontinuous of a function can $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0,1]$ be?

Problem 6.47. Take all possible pointwise limits of functions from $C[0,1]$, which we call B_1 (for Baire). Do the same with functions from B_1; call the new set of all possible pointwise limits B_2. Is it true that $B_1 \subsetneq B_2$? Some examples?

Problem 6.48 (Smith-Volterra-Cantor)**.** Construct sets which are homeomorphic to $\mathcal{C}$, but have positive measure. (Construct a Cantor-like set but with $\sum l_n < 1$.)

§7 Hausdorff-Banach-Tarski Paradox

The secret behind the paradox is different properties of isometry groups of $\mathbb{R}^2$ (denoted G_2) and $\mathbb{R}^3$ (denoted G_3).

Theorem 7.1
G_3 contains a free-subgroup, that is, a group isomorphic to $F_2 = \langle a, b\rangle$.

§7.1 Rotation matrices

In $\mathbb{R}^2$, the rotation matrix by φ radians counterclockwise is $\mathbf{r} = \begin{bmatrix} \sin\varphi & -\cos\varphi \\ \cos\varphi & \sin\varphi \end{bmatrix}$.

In $\mathbb{R}^3$, there are many pairs of 3×3 matrices in G_3 which generate a group isomorphic to F_2. Taking the compactification of F_2 induces the Cantor set.

> **Remark.** Many results in real analysis can be formulated in language which uses only the notion of measure zero. As a rule, these results can be proved also in the framework of measure zero only. Some instances of such principle are
>
> - Criterion for Riemann integrability.
> - Monotone functions are almost everywhere differentiable.
> - Almost every $x \in [0, 1]$ is normal in base 2.

> **Remark.** We discuss different types of *typicality* in the sense of Baire category.

Exercise 7.2. Can a shifted Cantor set $\mathcal{C} + x$ with $x \in \mathbb{R}$ consist solely of irrationals?

A more generalized question:

Exercise 7.3. Let $E \subseteq \mathbb{R}$, where $\lambda(E) = 0$. Is it true that for some $x \in \mathbb{R}$, we have $(E + x) \cap \mathbb{Q} = \emptyset$?

§8 Smith-Volterra-Cantor sets

Theorem 8.1 (Smith-Volterra-Cantor)

Let $a = \sum_{i=1}^{\infty} a_i$, where $a_i > 0$ and $a \in (0, 1]$. Delete the open intervals a_1, a_2, ..., akin to how we constructed the classical Cantor set. Let $\mathcal{K} := \bigcap_{i=1}^{\infty} \mathcal{C}_i$. Note that $\mu(\mathcal{K}) = 1 - a$. Yet, $\mathcal{K}$ is homeomorphic to the classical Cantor set, since each interval that we deleted is homeomorphic to the classical deleted interval; hence $\mathcal{K}$ is a Cantor set.

Theorem 8.2 (Steinhaus)

If $A \subseteq \mathbb{R}$ with $\mu(A) > 0$, then $A - A$ contains an interval.

Exercise 8.3. Is it necessarily true that $\mathcal{K} - \mathcal{K}$ contains an interval, for $a = 1$? (For $a < 1$, since $\mu(\mathcal{K}) = 1 - a > 0$, by Steinhaus, the problem is not of interest anymore.)

Solution. Not necessarily. Take $a_1 = 1$, then $\mathcal{K} = \{0, 1\}$, so $\mathcal{K} - \mathcal{K} = \{-1, 0, 1\}$, which does not contain an interval. □

Problem 8.4. What if we restrict $a_i < 1$?

Example 8.5

Take $\tilde{I}_n = \left(\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}\right) \cap \mathbb{Q}$. Then, $\bigcap_{k=1}^{\infty} I_k = \emptyset$.

Example 8.6

Take $\tilde{I}_n = \left(2 - \frac{1}{n}, 2 + \frac{1}{n}\right) \cap (\mathbb{R} \setminus \mathbb{Q})$. Then, $\bigcap_{k=1}^{\infty} I_k = \emptyset$.

Theorem 8.7 (Cantor et al.)

If $I_1 \supset I_2 \supset \dots$ is a sequence of closed nested (nonempty) intervals in $\mathbb{R}$, then

$$\bigcap_{k=1}^{\infty} I_k \neq \emptyset$$

By this, $\mathcal{K}$ is uncountable, since the process is itinerary for which $\mathcal{C}_i$ is closed for each iteration.

Exercise 8.8. Is $\mathcal{C} \times \mathcal{C}$ a Cantor set?

Yes, $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}$.

Exercise 8.9. Is $\{0, 1, 2, 3\}^{\mathbb{N}}$ a Cantor set?

Lengthen the sequence twice.

Exercise 8.10. Can you represent $\mathcal{C}$ as a finite disjoint union of n Cantor sets, for all n?

Proof. Take a left-right self-fractal. □

Exercise 8.11. Can you represent $\mathcal{C}$ as a countable disjoint union of Cantor sets?

Proof. Yes. □

Exercise 8.12. Can you represent $\mathcal{C}$ as an uncountable disjoint union of Cantor sets?

Proof. Yes. Take $C_i = \{(i, x)\} \subseteq \mathcal{C} \times \mathcal{C}$, and thus $\bigcup_i C_i \cong \mathcal{C}$ since $\mathcal{C} \cong \mathcal{C} \times \mathcal{C}$. Hence, we are done. □

Exercise 8.13. Can you get an uncountable number of disjoint uncountable sets, which all have measure zero, and get measure zero for its union?

Proof. Yes, the previous exercise immediately answers this. □

§9 Lebesgue measure in $\mathbb{R}$

Read *Princeton Lectures in Analysis III: Measure Theory, Integration, and Hilbert Spaces* by Stein.

Definition 9.1 (Outer measure)**.** Define the *outer measure* $\mu^* : 2^{\mathbb{R}} \to [0, \infty]$, which satisfies $\mu^*([a, b]) = b - a$ and $\mu^*((a, b)) = b - a$. One may define outer measure for all subsets of $\mathbb{R}$, that is, for a set $X \subseteq \mathbb{R}$, define $\mu^*(X) := \inf_{x \subseteq \bigcup_{i \in I} \mathcal{U}_i} \{\sum_{i \in I} \mu^*(\mathcal{U}_i)$, where $\mathcal{U}_i$ are open intervals and I is an index set (i.e., countable).

An outer measure μ^* satisfies the following properties:

1. $E \subseteq E' \implies \mu^*(E) \leq \mu^*(E')$ (monotonicity)

2. For countably many disjoint sets $\{E_i\}$,

$$\mu^*\left(\bigcup_i E_i\right) \leq \sum_i \mu^*(E_i)$$

   (countable sub-additivity)

> **Remark.** But we want something better, for example, we'd want
>
> $$\mu\left(\bigcup_i E_i\right) = \sum_i \mu(E_i)$$
>
> to hold for countably many disjoint sets $\{E_i\}$.

Definition 9.2. A set E is said to be *Lebesgue measurable* if for any $X \subseteq \mathbb{R}$,

$$\mu^*(E) = \mu^*(X \cap E) + \mu^*(X^{\complement} \cap E)$$

A *Lebesgue measure* $\mu : 2^{\mathbb{R}} \to [0, \infty]$ exists if and only if E is Lebesgue measurable, and in that case, $\mu(E) = \mu^*(E)$.

Theorem 9.3

All Borel sets are Lebesgue measurable.

There exists non-Borel sets that are Lebesgue measurable, since the cardinality of Lebesgue measurable sets is strictly larger than the cardinality of Borel sets.

Example 9.4 (Vitali set)

A *Vitali set* is a subset V of the interval $[0, 1]$ of real numbers such that, for each real number r, there is exactly one number $v \in V$ such that $v - r$ is a rational number.
Vitali sets are non-Borel but Lebesgue measurable.

A Lebesgue measure μ satisfies the following properties:

1. $E \subseteq E' \implies \mu(E) \leq \mu(E')$ (monotonicity)
2. For countably many disjoint sets $\{E_i\}$,

$$\mu\left(\bigcup_i E_i\right) = \sum_i \mu(E_i)$$

   (countable additivity)

3. $\mu(E + x_0) = \mu(E)$ for all x_0 (translation invariant)

Definition 9.5. A set E is *meager* if $E = \bigcup_{i=1}^{\infty} E_i$ for all i, and E_i is nowhere dense.

A set E is *nonmeager* if it is not meager.
A set E is called *comeager* (or a residual set) if $E^{\complement}$ is meager.

> **Remark.** To actually *construct* a Lebesgue measure, we need Carathéodory's extension theorem.

§9.1 Digression

Euler found a continuous fraction expansion for e^r where $r \in \mathbb{Q}$.

Definition 9.6. A *simple* continued fraction is of the form $\frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\dots}}}$ where the numerators are all 1.

There are three historically important ways of representing numbers and functions:

1. series
2. product
3. continuous fraction

Example 9.7 (Series representation)

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example 9.8 (Product representation, Euler)

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1-\frac{x}{k\pi}\right)\left(1+\frac{x}{k\pi}\right)$$

Example 9.9 (Basel problem)

Comparing coefficients of two different representations, we get the solution to the Basel problem:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 9.10 (Sophomore's dream)

$$\sum_{n=1}^{\infty} n^{-n} = \int_0^1 x^{-x}\,dx$$

Example 9.11 (Lindemann)

Lindemann showed that π^2 is transcendental, solving the famous Greek problem about the quadrature of the circle.

Example 9.12 (Apéri)

$$\sum \frac{1}{n^3} \notin \mathbb{Q}$$

Problem 9.13. Why is the set of all base 2 normal numbers in $(0, 1]$ measurable?

Theorem 9.14 (Cassels)
Almost every $x \in \mathcal{C}$, with respect to the natural probability measure on $\mathcal{C}$, is base 2 normal.

We define a measure on $\mathcal{C}$ quite differently than what we usually do (the Borel measure). We give the first left half interval measure $\frac{1}{2}$ and the right half interval measure $\frac{1}{2}$ as well. Then, for the second iteration, we assign measure $\frac{1}{4}$ for each interval.

Problem 9.15. Show that the set of normal numbers is uncountable, only using $c = 0.1234567891011\ldots$ is normal.

Proof. One may delete any single digit, which still gives us a normal number since normality is a symplectic property. We may pick a n^2th digit, and either delete it or not. Then, we have uncountably many normal numbers, hence we are done. □

Let $f_n : [0, 1] \to \mathbb{R}$ be continuous functions. Assume that pointwise limit exists, i.e.,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Exercise 9.16. Is f measurable?

Exercise 9.17. Is f_n measurable, given that f_n is uniformly bounded?

Let $\Omega = \{0, 1\}^{\mathbb{N}} = C_{1;0} \cup C_{1;1}$ where $C_{1;0} = \{x \in \Omega : x_1 = 0\}$ and $C_{1;1} = \{x \in \Omega : x_1 = 1\}$.

Definition 9.18. Two measure spaces $\overline{X}_1 = (X_1, \mathfrak{B}_1, \mu_1)$ and $\overline{X}_2 = (X_2, \mathfrak{B}_2, \mu_2)$ are isomorphic if there exists a "one-to-one almost everywhere" measure-preserving mapping φ between $\overline{X}_1$ and $\overline{X}_2$ such that $X_1 \xrightarrow{\varphi} X_2$ and $\forall A \in \mathfrak{B}_1$, $\mu_1(A) = \mu_2(\varphi(A))$.

That is, $\exists N_1 \in \mathfrak{B}_1$, $\mu_1(X_1 \setminus N_1) = 1$ and $\exists N_2 \in \mathfrak{B}_2$, $\mu_2(X_2 \setminus N_2) = 1$ such that $\varphi : X_1 \setminus N_1 \to X_2 \setminus N_2$ is a one-to-one bimeasurable bijection which preserves measure.

Remark. We write N_1 and N_2 for negligible sets.

Exercise 9.19. Prove that $[0, 1] \cong [0, 1] \times [0, 1] \cong \{0, 1\}^{\mathbb{N}}$ as measure spaces.

Definition 9.20. Denote $C_{n_1,n_2,\ldots,n_k;\, i_1,i_2,\ldots,i_k}$ as all $x \in \Omega$ which have i_t at coordinate n_t, with $n_1 < n_2 < \cdots < n_k$ and $i_j \in \{0, 1\}$. Then, $C_{n_1,n_2,\ldots,n_k;\, i_1,i_2,\ldots,i_k}$ generates a σ-algebra of subsets of Ω.

We have $\mu(C_{1;0}) = \mu(C_{1;1}) = \frac{1}{2}$.

Consider the set $C_{17,21;\,0,1}$, that is, all $x \in \Omega$ which have 0 at coordinate 17 and 1 at coordinate 19. Then, $\mu(C_{17,21;\,0,1}) = \frac{1}{4}$.

Exercise 9.21. There are uncountably many measures on the symbolic space of $\{0,1\}^{\mathbb{N}}$. (Think of an unfair coin, with probability p and q.)

§10 Hamel basis

Definition 10.1. A *Hamel base* in $\mathbb{R}$ is a base in the vector space $\mathbb{R}_{\mathbb{Q}}$.

Example 10.2

$\dim \mathbb{R}_{\mathbb{R}} = 1$. $\dim \mathbb{R}_{\mathbb{Q}}$ is not finite.

Theorem 10.3

Any vector space has a base.

Theorem 10.4

Any field F is a vector space over its fixed subfield F_0. Hence, $ax+by$ are well-defined $\forall x,\, y \in F$ and $\forall a,\, b \in F_0$.

Theorem 10.5 (Hamel basis)

If $H \subseteq \mathbb{R}$ is a Hamel base, then $\forall x \in \mathbb{R}$, x can e uniquely written as $x = \sum_i a_i h_i$ where $h_i \in H$ and $a_i \in \mathbb{Q}$.

Exercise 10.6. Prove that $|H| = |\mathbb{R}|$.

Exercise 10.7. Prove that $\{\sqrt{p} \mid p \in \mathbb{P}\}$ are $\mathbb{Q}$-independent.

§10.1 Cauchy functional equation

Definition 10.8 (Cauchy FE)**.** Given $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x,\, y \in \mathbb{R}$. We call this a Cauchy functional equation on $\mathbb{R}$.

Exercise 10.9. If f is continuous, then $f(x) = cx$ for some $c \in \mathbb{R}$.

Exercise 10.10. If f is assumed to be measurable, then $\exists c$ such that $f(x) = cx$.

Exercise 10.11. Prove that there exists uncountably many solutions to the Cauchy functional equation.

§11 Young tableaux

Definition 11.1 (Alphabet)**.** Let $w = u_1 \ldots u_n$ where $u_i \in \mathbb{Z}_{>0}$.

We have a monoid (free semigroup on infinitely many generators) M with $\emptyset \in M$. We have the operation concatenation, that is, for $w_1 = u_1 \ldots u_n$ and $w_2 = v_1 \ldots v_m$, we have $w_1 w_2 = u_1 \ldots u_n v_1 \ldots v_m$. There are two transformations $K' : xyz \to xzy$ if $z < x \leq y$, and $K'' : xyz \to yxz$ if $x \leq z < y$.

Definition 11.2 (Knuth equivalence)**.** We call two words w_1, w_2 *Knuth equivalent* if one may obtain $w_1 \rightsquigarrow w_2$ via K' and K''. We denote such equivalence relations as $w \equiv v$.

Thus, we have $w_1 \equiv w_2$ and $v_1 \equiv v_2$ then $w_1 v_1 \equiv w_1 v_2 \equiv w_2 v_2 \equiv w_2 v_1$.
We may take the plactic monoid $M = {}^{F}/_{R}$.

Definition 11.3 (Young diagram)**.** A *Young diagram* (or sometimes called a *Ferrers diagram*) is a left-aligned collection of boxes with weakly decreasing rows. Formally, a Young diagram is $\{\lambda \vdash n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\}$ where $\lambda \vdash n$ means λ varies over all partitions of n, which is equivalent to a Young diagram with n boxes.

Example 11.4

An example of a Young diagram is $(4, 4, 2, 2)$.

Definition 11.5 (Young tableaux)**.** A *Young tableaux* is a "filling" of a Young diagram with positive integers such that the numbers are weakly increasing along rows and strictly increasing down columns.

Remark. Young tableaux were first introduced by Alfred Young in 1900.

Definition 11.6 ("Bumping")**.** Define a *bumping* (row-insertion / Schensted operation) of a tableau as follows. Given $x \in \mathbb{Z}_{>0}$ and T, we "insert" x (denoted as $T \leftarrow x$) and get a different tableau, performing either one of the following two operations:

1. If x is greater than equal to all of the first row, then place x on the end.

2. Otherwise, place x as far to the right as possible, and then "bump" the entry of that box to the next row.

Define the *word* of a tableau as the sequence of entries read left to right, **bottom to top** (this gives the uniqueness). Note the natural isomorphism between the Knuth equivalence we defined and the entries of a tableau.

Theorem 11.7

In each Knuth equivalence class, there exists a unique word that corresponds to a tableau.

Let M_m be a monoid, and consider $\mathbb{Z}[M_m]$. The morphism $\varphi : \mathbb{Z}[M_m] \to \mathbb{Z}[x_1, \ldots, x_m]$. We have $x^T = \prod_{i=1}^m x_i^{c_i(T)}$. Then, $\varphi\left(\sum_{T \text{ shape } \lambda} T\right) = S_\lambda(x_1, \ldots, x_m)$, where C_λ is the number of i's in T, and $S_\lambda(x_1, \ldots, x_m) \cdot h_p(x_1, \ldots, x_m) = \sum S_M(x_1, \ldots, x_m)$.

Given a word in $\mathbb{Z}_{>0}$, we can get a tableau associated to this word, along with a "recording" tableau (of the same shape).

Theorem 11.8 (Robinson correspondence)

There exists a one-to-one correspondence between words of length n with entries $\{1, 2, \ldots, n\}$ and (P, Q) of standard tableau of same shape with n boxes.

Theorem 11.9 (Tableaux identity)
If f^λ is the number of standard tableaux of shape λ, then

$$n! = \sum_{\lambda \vdash n} \left(f^\lambda\right)^2$$

Theorem 11.10 (Robinson–Schensted correspondence)
There exists a one-to-one correspondence between words of length n with entries in $[m]$ and (P, Q) with same shape, where Q is standard and P has entries in $[m]$.

Theorem 11.11 (Robinson–Schensted–Knuth correspondence)
There exists a one-to-one correspondence between two rowed arrays in lexicographical order with row length n and (P, Q) which is a tableau of same shape with n boxes.

We can reconstruct a tableau from (P, Q) as follows. Given an order 2 row array, on the bottom row, we get a T corresponding to the bottom row. Then, fill the recording tableau with entries in the top row. Repeat this procedure until we don't have any entries left.

Theorem 11.12 (Cauchy)

$$\prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1}{1 - x_i y_j} = \sum_{\lambda} S_\lambda(x_1, \dots, x_n) S_\lambda(y_1, \dots, y_m)$$

Definition 11.13. A probability space is a measure space $(X, \mathfrak{B}, \mu)$ with $\mu(X) = 1$.

Definition 11.14. For a transformation $T : X \to X$, we say T *preserves measure* when $\forall A \in \mathfrak{B}$, $\mu(T^{-1}(A)) = \mu(A)$.

Example 11.15
Consider $T(x) = 2x \bmod 1$.

Definition 11.16. For two measure spaces $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$, we say a mapping $f : X_1 \to X_2$ is *measure-preserving* if $\exists Y_i \subseteq X_i$ such that $\mu_i(X_i \setminus Y_i) = 0$ and $\mu_1(Y_i) = \mu_2(f(Y_i))$.

Definition 11.17 (Outer measure on $\mathbb{R}^d$)**.** Take any set $E \in \mathcal{P}(\mathbb{R}^d)$. Then, define the outer measure of E to be

$$\mu^*(E) := \inf_{E \subseteq \bigcup_{i=1}^\infty Q_i} \sum_{i=1}^{\infty} |Q_i|$$

where $\{Q_i\}_{i=1}^\infty$ is any countable collection of closed cubes.

Remark. Note that μ^* does not have the "ideal" condition for a measure. In fact, it doesn't even have finite additivity.

Definition 11.18 (Lebesgue measurable sets)**.** A set E is called *Lebesgue measurable* if $\forall \varepsilon > 0$, there exists an open set $\mathcal{O} \subseteq \mathbb{R}^d$ with $E \subseteq \mathcal{O}$ such that $\mu^*(\mathcal{O} \setminus E) < \varepsilon$.

Theorem 11.19 (Criterion for Lebesgue measurability)

A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable* if and only if E differs from a G_δ (countable intersection of open sets) or F_σ (countable union of closed sets) set with a set of Lebesgue measure zero.

Theorem 11.20

$f : \mathbb{R}^d \to \mathbb{R}$ is called to be measurable if $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a))$ is measurable. Equivalently, f is measurable if for all open sets $\mathcal{O} \subseteq \mathbb{R}$, $f^{-1}(\mathcal{O})$ is measurable.

Exercise 11.21. Prove that the following are properties of measurable functions:

1. If f is measurable, then f^k is measurable.
2. If f and g are measurable, then $f + g$ is measurable.
3. If f and g are measurable, then $f \cdot g$ is measurable. (Pointwise multiplication)
4. If $\{f_n\}$ is a sequence of measurable functions, then $\limsup f_n(x)$, $\liminf f_n(x)$, $\sup f_n(x)$, $\inf f_n(x)$ are measurable.
5. If $\{f_n\}$ is a sequence of measurable functions, then if $\lim_{n\to\infty} f_n(x) = f(x)$ exists, then f is also measurable.

Exercise 11.22. Prove that the following are properties of continuous functions:

1. If f is continuous, then f^k is continuous.
2. If f and g are continuous, then $f + g$ is continuous.
3. If f and g are continuous, then $f \cdot g$ is continuous. (Pointwise multiplication)
4. If $\{f_n\}$ is a sequence of uniformly continuous functions, then $\lim_{n\to\infty} f_n(x) = f(x)$ exists, and f is also continuous.

§12 Cesàro limits

Definition 12.1. If $\frac{1}{N}\sum_{i=1}^{N} a_i \to a$, we say that a is the *Cesàro limit* of $(a_i)_{i=1}^{\infty}$, which we denote as $\text{clim}(a_i) = a$.

Example 12.2

Consider the sequence $a_i = \{1, -1, 1, \dots\}$.

Taking the Cesàro limit, we have $\text{clim}(a_i) = 0$.

1. $\text{clim}(a_i + b_i) = \text{clim}(a_i) + \text{clim}(b_i)$
2. $\text{clim}(a_i \cdot b_i) \neq \text{clim}(a_i) \cdot \text{clim}(b_i)$

Example 12.3 (Szemerédi)

$$\overline{d}(A) := \limsup_{N\to\infty} \frac{|A \cap \{1,2,\dots,N\}|}{N} = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_A(n).$$

Example 12.4 (Normal sequences)

$x = \sum_{i=1}^{\infty} \frac{t_i}{2^i}$ is base t normal if $d_k := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_k(t_i) = \frac{1}{t}$ for all $k \in \{0,1,2,\dots,t-1\}$

where $\mathbb{1}_k(i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$

Exercise 12.5. Prove that arbitrarily permuting the elements of a sequence can "kill" positive density.

Exercise 12.6. Prove that $x \in [0,1]$ is normal in base 2 if and only if the sequence $2^n x \bmod 1$ is uniformly distributed in $[0,1]$.

Definition 12.7. A sequence $(x_n) \subseteq [0,1]$ is *uniformly distributed* if $\forall a, b$ such that $0 \le a < b \le 1$,

$$\lim_{N\to\infty} \frac{|\{1 \le n \le N : x_n \in (a,b)\}|}{N} = b - a$$

or equivalently,

$$\forall f \in C[0,1], \quad \frac{1}{N}\sum f(x_n) \to \int_0^1 f\,dx$$

which is also equivalent to

$$\lim_{N\to\infty} \frac{1}{N}\sum \mathbb{1}_{(a,b)}^{(x_n)} \to \int_0^1 \mathbb{1}_{(a,b)}^{(x)}\,dx = b - a$$

Theorem 12.8

Almost every $(x_n) \subseteq \mathbb{R}$ is uniformly distributed mod 1.

Natural examples of uniformly distributed sequences mod 1 are:

Example 12.9 (Weyl)

$\{n\alpha\}$ and $\{n^2\alpha\}$, where $\alpha \notin \mathbb{Q}$.

Exercise 12.10. Prove that $\{n\alpha\}$ is uniformly distributed in $[0,1]$, where $\alpha \notin \mathbb{Q}$.

Example 12.11 (Fejer)

$\{n^c\}$, where $c > 0$, $c \notin \mathbb{N}$.

Theorem 12.12 (Weierstrass' approximation theorem)
$\forall f \in C[0,1]$, $\forall \varepsilon > 0$, there exists a polynomial $g(x) \in \mathbb{R}[x]$ such that

$$\max_{x\in[0,1]} |f(x) - g(x)| < \varepsilon$$

Definition 12.13. A metric space (X, d) is *separable* if there exists a countable dense subset in X, i.e., $S \subseteq X$, S countable, and $\overline{S} = X$.

Example 12.14
A nonseparable space. Let S be an uncountable set equipped with a discrete metric.

Exercise 12.15. Is $L^\infty(\mathbb{R})$ separable?

Example 12.16
$C[0,1]$ is separable, since every function can be approximated by polynomials by Weierstrass' approximation theorem, which can again be approximated by rational polynomials.

§13 Fourier series

Remark. We want to know when and how $\sum_1^\infty (a_n \sin(nx) + b_n \cos(nx)) = f(x) \in C[0,1]$ holds.

Ideally, we want $\sum_{n=0}^N (a_n \sin(n) + b_n \cos(nx)) = \sigma_N(f) \to f(x)$ uniformly for all x. However, it is not the case.

Theorem 13.1 (Fejer)
Let $\sigma_N(f) := \sum_{n=0}^N (a_n(f)\sin(nx) + b_n(f)\cos(nx))$ for $x \in [-\pi, \pi]$, then

$$\frac{\sigma_1(f) + \sigma_2(f) + \cdots + \sigma_N(f)}{N} \rightrightarrows f(x)$$

uniformly, in the norm of $C[-\pi, \pi]$.

Theorem 13.2 (Trigonometric form of Weierstrass)
For any $f \in C[-\pi, \pi]$, $\forall \varepsilon > 0$, there exists a trigonometric polynomial $T(x)$ such that $\max_{x\in[-\pi,\pi]} |f(x) - T(x)| < \varepsilon$.

Exercise 13.3. Prove that Fejer's theorem implies Weierstrass' approximation theorem.

Example 13.4

Note that $\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$ on $[-\pi, \pi]$. Thus, $\frac{x^2}{4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$, and plugging in $x = \frac{\pi}{2}$, we get a proof for the Basel problem.

Theorem 13.5

If $f_n(x) \in C[0,1]$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists $\forall x \in [0,1]$, then $f(x)$ has many points of continuity. More precisely, the set of points of continuity of f is a dense G_δ set (countable intersection of open sets).

δ means intersection and σ means union.
$G_{\sigma\delta\sigma}$ is the countable union of countable intersection of countable union of open sets.

Theorem 13.6

If $f_n \in C[0,1]$ and $f_n \to f$ in $C[0,1]$, then $f \in C[0,1]$.

Definition 13.7 (Baire class)**.** Baire class 0: continuous functions $C[0,1]$. Baire class 1: pointwise limits of functions from Baire class 0. Baire class 2: pointwise limits of functions from Baire class 1. ...

Example 13.8

Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$, then f is in Baire class 2.

§14 Invariant subspace problem

> **Remark.** Every linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvalue and its associated eigenvector.

Definition 14.1. $\exists x \neq 0$ such that $Tx = \lambda x$ for some λ. Let $V = \mathrm{Span}(x)$, then $T(V) \subseteq V$, where we call V an *invariant subspace.*

Problem 14.2. Given a Banach space X, a bounded linear operator T on X, does there exist a closed invariant subspace, i.e., for $U \subsetneq X$, $T(U) \subseteq U$?

Definition 14.3. A *Banach space* X is a vector space over $\mathbb{C}$, with $\|\cdot\|_X$ that is complete.

Example 14.4

ℓ^p spaces (sequence spaces) are Banach spaces.

Example 14.5

L^p spaces (function spaces) are Banach spaces as well.

Definition 14.6. A *linear operator* T on X is a linear mapping $X \to X$ such that $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$.

Definition 14.7. A linear operator T is bounded if

$$\|T\|_{op} = \sup_{x \in X,\, \|x\|=1} \|Tx\| < \infty$$

Example 14.8

Here are examples of unbounded operators: $(Tx_n) = (2^n x_n)$ in $\ell^1(\mathbb{N})$.

Theorem 14.9

A linear operator T is bounded $\iff$ T is continuous.

Proposition 14.10

If X is non-separable, then it has a closed invariant subspace $\forall T$.

Proof. Consider the closure of the span of $\{T^n x\}$, which is separable. □

Definition 14.11. A linear operator T is compact if $\overline{T(B_1)}$ is compact.

Example 14.12

Linear operators with its image having finite rank are compact.

Theorem 14.13 (Schauder, 1930)

If X is a Banach space and F is a continuous linear operator, satisfying $F(C) \subseteq K \subseteq C$, where K is a compact set and C is a convex set, then there exists a fixed point, i.e., $F(X) = X$ in C.

Theorem 14.14 (Lomonosov, 1973)

If $T \subset B(X, X)$ (i.e., any Banach space) is compact, then it has a closed invariant subspace.

Proof. Assume FTSOC that T has no eigenvector. $\forall y \in X$, consider $M_y = \{Sy : ST = TS\}$, which is closed (exercise: show that it forms an algebra). It suffices to prove that $M_y \neq X$ for some y. Suppose $M_y = X$ $\forall y$. Choose $B_1(x_0) \not\ni 0$.

Claim 14.15 — For all y, there exists a neighborhood W of y such that $S(W) \subset B$ for some S.

Proof. For all S with $ST = TS$ and $Sy \in B$ for some y, $\exists W_S$ such that $S(W_S) \subseteq B$ and $\{W_S\}$ is an open cover for X. Then, $\{W_S\}$ covers $\overline{T(B)}$, but since T is a compact operator and B is a Banach space, we may find a finite subcover that covers $\overline{T(B)}$, which means $\{W_1, \ldots, W_n\}$ covers $\overline{T(B)}$.

Define $\Phi(y) = \sum_{i=1}^{n} \frac{q_i(y)}{q(y)} S_i(y)$, where $q_i(y) = \max(0, 1 - \|S_i(y) - x_0\|)$ and $q(y) = \sum_i q_i(y)$, where x_0 is the center of B. Then, since Φ is continuous, $\Phi(\overline{T(B)}) \subset B$ is also compact, thus by Schauder fixed point theorem, $\Phi \circ T(B) \subseteq$ compact set $\subseteq B$.

Now, $\Phi \circ T(x^*) = x^*$, so $\sum(\frac{q_i(x^*)}{q(x^*)}) S_i \circ Tx = x$, which has a finite dimensional eigenspace due to spectral theory, contradiction. ■

□

Theorem 14.16 (Per Enflo)

Per Enflo claims to have solved the general problem in Hilbert spaces.

§15 Measures

Read *Measure & Category* by Oxtoby, and *Real Analysis* by Royden. (NOT Royden & Fitzpatrick.)

Theorem 15.1 (Lebesgue's "points of density" theorem)

Given $A \subset \mathbb{R}$ with $\mu(A) > 0$, then for almost every $x \in A$,

$$\lim_{\varepsilon \to 0} \frac{\mu(A \cap (x - \varepsilon, x + \varepsilon))}{2\varepsilon} = 1$$

Corollary 15.2 (Steinhaus)

If $A \subset \mathbb{R}$, $\mu(A) > 0$, then $A - A \supset (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Problem 15.3. $\forall \varepsilon > 0$, is there E such that $\mu(E \cap I) = \frac{1}{2}\mu(I)$ for all $I = (a, b)$ where $|b - a| < \varepsilon$.

No, since $\exists \varepsilon > 0$ such that

$$\frac{\mu(A \cap (x - \varepsilon, x + \varepsilon))}{\mu(x - \varepsilon, x + \varepsilon)} = 1$$

which is directly implied by Lebesgue's "points of density" theorem.

Exercise 15.4. True or false: If $A \subset \mathbb{R}$ and $\mu(A) > 0$, then A contains a Cantor set.

Lemma 15.5 (Regularity of Lebesgue measure)

If $A \subset [0, 1]$, then A contains a compact subset K such that $|\mu(A) - \mu(K)| < \varepsilon$.

Lemma 15.6 (Dual of regularity of Lebesgue measure)

If $A \subset [0, 1]$, then there exists an open set $\mathcal{O}$ such that $A \subset \mathcal{O}$ and $|\mu(A) - \mu(\mathcal{O})| < \varepsilon$.

Exercise 15.7. True or false: If $A \subseteq \mathbb{R}$ is uncountable and compact, then A contains a Cantor set.

Exercise 15.8 (Stronger version)**.** True or false: If $A \subset \mathbb{R}$ and $\mu(A) > 0$, then A contains a Cantor set of positive measure.

Lemma 15.9

If $\mu(A) = a > 0$, then $\exists A_1 \subset A$ such that $\mu(A_1) = \frac{a}{2}$. Moreover, for any $t \in [0, a]$, $\exists A_t \subset A$ such that $\mu(A_t) = t$. That is, the range of μ on $\{A \cap C \mid C \in \mathfrak{B}\}$ is $[0, \mu(A)]$.

Exercise 15.10. Is there a fat Cantor set in $\mathbb{R} \setminus \mathbb{Q}$?

First, it is easy to create a Cantor subset in $\mathbb{R} \setminus \mathbb{Q}$.
Take $\mathbb{Q} \cap [-\sqrt{2}, \sqrt{2}]$, which we let to be $r_1, r_2, \ldots$.
Then, take ε_1 small enough so that $(r_1 - \varepsilon_1, r_1 + \varepsilon_1)$ fits in the interval $[-\sqrt{2}, \sqrt{2}]$; take another interval centered at a rational point disjoint to all previous ones, and repeat. Now, just take these intervals so that the sum of their measures is less than $\mu([-\sqrt{2}, \sqrt{2}])$, then we have a fat Cantor set.

Definition 15.11. Define a general measure $\mu_f((a, b)) := f(b) - f(a)$ for any monotone (not necessarily continuous) function f.

> **Remark.** Why do we specifically care about Lebesgue measure? Because $f(x) = x$ is the unique function that satisfies translation invariance of measure, that is, $\mu(A) = \mu(A + t)$ for all $t \in \mathbb{R}$. (Proof: Cauchy functional equation directly implies that $f(x) = x + c$.)

§15.1 Three Littlewood's principles

The following principles are *philosophical* principles, not mathematical ones, formulated by J. E. Littlewood.

- Sets of positive measure are "locally" intervals. (cf. Lebesgue's "points of density" theorem)
- Measurable functions are "almost" continuous. (cf. Lusin's theorem)
- Pointwise convergence for a sequence of measurable functions is "almost" uniform convergence (for $\{f_n\}$ defined on $[0, 1]$). (cf. Egorov's theorem)

Theorem 15.12 (Egorov's theorem)

For (X, A, μ) where $X \subseteq \mathbb{R}^n$, let $\{f_n\}$ be a sequence of measurable functions $f_n : \mathbb{R}^n \to \mathbb{R}$, with $\{f_n(x)\} \to f(x)$ almost everywhere. Then, $\forall \varepsilon > 0$, one may find some closed set $F_\varepsilon \subseteq X$ such that $\mu(X \setminus F_\varepsilon) < \varepsilon$ and $\{f_n\} \to \{f\}$ uniformly on F_ε, that is, $\limsup_{x \in F_\varepsilon} |f_n(x) - f(x)| < \varepsilon$.

Proof. For each n, $k \geq 0$, define $E_k^n := \{x \in X : |f_j(x) - f(x)| < \frac{1}{n} \text{ for all } j, k\}$.

Then, for $\varepsilon > 0$, one may always find some N such that $\overline{A_\varepsilon} := \bigcap_{n\geq N} E_k^n$ and $\mu(X \setminus \overline{A_\varepsilon}) < \varepsilon/2$. [Exercise: show that $f_n \to f$ uniformly on $\overline{A_\varepsilon}$.] Now, choose a closed subset $F_\varepsilon \subseteq \overline{A_\varepsilon}$ such that $\mu(\overline{A_\varepsilon} \setminus F_\varepsilon) < \varepsilon/2$, then $\{f_n\} \to f$ on F_ε, where F_i is a closed subset, with $\mu(X \setminus F_\varepsilon) < \mu(X \setminus \overline{A_\varepsilon}) + \mu(\overline{A_\varepsilon} \setminus F_\varepsilon) < \varepsilon$, so we are done. □

Theorem 15.13 (Lusin's theorem)

Let $X \subseteq \mathbb{R}^n$ be a measurable set, f be measurable and finite valued on X, that is, $f(x) < \infty$ for all $x \in X$. Then, $\forall \varepsilon > 0$, $\exists F_\varepsilon \subseteq X$ closed such that $F_\varepsilon \subseteq X$ and $\mu(X \setminus F_\varepsilon) < \varepsilon$ such that $f|_{F_\varepsilon}$ is continuous.

Exercise 15.14. Can you "modify" a function on a measure zero set to make it continuous?

Exercise 15.15. If $A \subset \mathbb{R}$, $\mu(A) > 0$, then A contains an affine image of any finite set F, that is, $F = \{x_1, \ldots, x_n\} \implies A \supset aF + b$ for some $a \neq 0$, $b \in \mathbb{R}$.

> **Remark.** The groups of rigid motions $G(\mathbb{R}^2)$ and $G(\mathbb{R}^3)$ have an important distinction: $G(\mathbb{R}^2)$ is amenable, and $G(\mathbb{R}^3)$ is not amenable. This is the background of the Hausdorff-Banach-Tarski paradox.

Definition 15.16. A countable group G is *amenable* if there exists a finitely additive probability measure on $\mathcal{P}(G)$ such that $\forall g \in G$ and $\forall A \subset G$, $\mu(A) = \mu(Ag) = \mu(gA)$, where $gA = \{gx \mid x \in A\}$.

> **Remark.** The advantages are that every group in the discrete topology is measurable, and that you get a probability measure. One disadvantage is that the measure is only finitely additive.

Any large set $A \subset \mathbb{R}$ is combinatorially rich, i.e., it contains an affine image of any finite set $F \subset \mathbb{R}$.

Exercise 15.17. Prove that for any $E \subset \mathbb{Z}$ such that $\overline{d}(E) > 0$, E is combinatorially rich, i.e., it contains an affine image of any finite set $F \subset \mathbb{Z}$.

Proof. By Szemerédi's theorem, we may "insert" F into E by translation and scaling, so we are done. □

Exercise 15.18. Are there sets of zero measure in $\mathbb{R}$ that are combinatorially rich?

Problem 15.19. Is the Cantor set combinatorially rich?

Exercise 15.20. Prove that the Lebesgue measure on $\mathbb{R}^n$ is invariant with respect to all rigid motions, which is a unique property up to normalization.

Problem 15.21. Give a counterexample for sets A, $B \subset \mathbb{R}$ such that both A and B are Lebesgue measurable, yet $A + B$ is not Lebesgue measurable.

Definition 15.22. $L^2[0,1]$ are the classes of equivalent measurable functions $f : [0,1] \to \mathbb{R}$ which are square-integrable, i.e., $\int_0^1 |f|^2 < \infty$. Then, we have the norm $\|f\|_{L^p} := (\int_X |f|^p d\mu)^P \frac{1}{p}$, and $\|f\|_{L^p}$ is an equivalence class:

- $\|f\|_{L^p} = 0 \iff f \equiv 0$ (($\implies$) by definition)
- $\|\lambda f\|_{L^p} = |\lambda| \|f\|_{L^p}$ (homogeneity)
- $\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$ for $p \in [1, \infty)$ (triangle inequality)

The third inequality is Minkowski's inequality, which can be shown by weighted AM-GM and Hölder.

The formal statements of those inequalities are as follows:

Lemma 15.23 (Weighted AM-GM)
For all $\lambda \in [0, 1]$, $A^\lambda B^{1-\lambda} \le \lambda A + (1 - \lambda)B$.

Lemma 15.24 (Hölder)
For $\frac{1}{p} + \frac{1}{q} = 1$, we have $\|fg\|_{L^1} \le \|f\|_{L^p} \cdot \|g\|_{L^q}$.

Lemma 15.25 (Minkowski)
For $p \in [1, \infty)$, $\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$.

Definition 15.26. $\ell^2 = \{x = (x_1, x_2, \dots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$.

Exercise 15.27. Prove that $d(f, g) = \sqrt{\int_0^1 |f - g|^2 dx}$ is a metric.

Definition 15.28. Define $L^p(X, \mathfrak{B}, \mu)$ to be the set of functions f with

$$\int_X |f|^p d\mu < \infty$$

where μ is σ-finite (meaning it has countable additivity).

> **Remark.** If we allow μ to be infinite, then $\ell^p = L^p(\mathbb{N}, \mathfrak{B}, \text{counting measure})$.

Let $f : C[-\pi, \pi]$ with $f \in L^2[-\pi, \pi]$. Then, we have the following analogues between L^2 spaces and ℓ^2 spaces. . . .

- Metric of L^2: $d(f_1, f_2) := \sqrt{\int_\pi^\pi |f_1 - f_2|^2} = \|f_1 - f_2\|_{L^2}$.
- Inner product of L^2: $\langle f_1, f_2 \rangle := \int_{-\pi}^{\pi} f_1 \overline{f_2}$.
- Orthogonal basis of $L^2[-\pi, \pi]$ is given by $\{1, \cos(nx), \sin(nx)\}$ for $n \in \mathbb{N}$, that is, any $f \in L^2$ can be expanded into infinite convergent series of the form $\sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx)$.

> **Remark.** Functional analysis is linear algebra in infinite-dimensional spaces.

Theorem 15.29

In $L^2[-\pi,\pi]$, $\|\sigma_N(f) - f\|_{L^2} \to 0$ as $N \to \infty$.

Problem 15.30. Is the analogue of the above theorem true in $C[-\pi,\pi]$?

Definition 15.31. An extreme point x of a convex set C is a point which has no nontrivial representations of the form $x = \alpha x_1 + (1-\alpha)x_2$ where $\alpha \in (0,1)$, $x_1, x_2 \in C$, and $x_1 \neq x_2$.

Compact convex sets (convex bodies, usually assumed to have nontrivial interior) in $\mathbb{R}^n$ have extreme points.

The set of extreme points of nontrivial convex body is of 2D measure zero.

Exercise 15.32. Why is there always an extreme point?

> **Remark.** The extreme points form a "basis" of a convex set.

Exercise 15.33. Prove that Lebesgue's density theorem implies Steinhaus.

[Hint: Alternative formulation of Steinhaus is $\mu(A \cap A - t) > 0$ for all small enough t, that is, $\lim_{t\to 0} \mu(A \cap A - t) = \mu(A)$.]

Exercise 15.34. Is there a nonmeasurable set whose set of differences contain a nontrivial interval?

Cauchy functional equations give the self-homomorphisms of $\mathbb{R}$.

Exercise 15.35. What are all the self-homomorphisms of $\mathbb{C}$?

Exercise 15.36. Prove that there exists a set A such that A is measurable, but $A + A$ is nonmeasurable.

Lemma 15.37

The classical Cantor set contains a basis of $\mathbb{R}_{\mathbb{Q}}$.

Proof. First, note that since $\mathcal{C} + \mathcal{C} = [0,2]$, $\mathcal{C}$ is a spanning set of $\mathbb{R}_{\mathbb{Q}}$. Second, any spanning set contains a basis. ■

Proof. Let $H \subseteq \mathcal{C}$ be a Hamel base of $\mathbb{R}_{\mathbb{Q}}$. Note that $\mu(H) = 0$. Let $\Gamma_1 = QH = \{rh : r \in \mathbb{Q}, h \in H\}$. Note that $\mu(\Gamma_1) = 0$, since $\Gamma_1 = \bigcup_r rH$ where a countable union of measure zero sets is of measure zero. Let $\Gamma_2 = \Gamma_1 + \Gamma_1$. Note that $\mu(\Gamma_2) = 0$. Inductively, define $\Gamma_n := \Gamma_{n-1} + \Gamma_{n-1}$. Suppose Γ_n are all measurable, then $\mu(\Gamma_n) = 0$.

But then, since every element in $\mathbb{R}$ is representable as a finite linear combination of H, thus $\mathbb{R} = \bigcup_{n=1}^{\infty} \Gamma_n$.

If $\mu(\Gamma_n) = 0$ for all $n \in \mathbb{N}$, then $\mu(\bigcup_{n=1}^{\infty} \Gamma_n) = 0$, that is, $\mu(\mathbb{R}) = 0$, contradiction.

If $\exists n \in \mathbb{N}$ such that $\mu(\Gamma_n) > 0$, then by Steinhaus, $\Gamma_{n+1} = \Gamma_n + \Gamma_n = \mathbb{R}$, but then $\mathbb{R}$ is finitely generated, which means $\dim \mathbb{R}_{\mathbb{Q}} < \infty$, contradiction to the fact that $\dim \mathbb{R}_{\mathbb{Q}} = \infty$.

Hence, $\exists n \in \mathbb{N}$ such that Γ_n is measurable yet $\Gamma_n + \Gamma_n$ is not measurable, which is exactly what we wanted to show. □

Remark. Read the principle of condensation of singularities.

Definition 15.38. A Borel measure ν on $[0,1]$ is *non-atomic*, if $\forall A \in \mathfrak{B}$ with $\nu(A) > 0$, $\exists \tilde{A} \subset A$ such that $0 < \nu(\tilde{A}) < \nu(A)$.

Exercise 15.39. Assume that ν is a non-atomic probability measure on $\mathfrak{B}([0,1])$. Then, $\{\nu(A), A \in \mathfrak{B}\} = [0,1]$.

Theorem 15.40 (Lyapunov's theorem about vector measures)

Assume that $\nu_1, \nu_2, \ldots, \nu_n$ are non-atomic probability measures on $\mathfrak{B}([0,1])$. Then, the range of $(\nu_1, \nu_2, \ldots, \nu_n)$, denoted as $k = \{\nu_1(A), \nu_2(A), \ldots, \nu_n(A) : A \in \mathfrak{B}\}$, is compact and convex (in $\mathbb{R}^n$).

The ranges of such vector measures are called *zonoids*.