Graph Theory

Jiwu Jang

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This is a note on a series of lectures on graph theory, given by Prof. Matthew Stone[1].

§1 Relations

Definition 1.1. A *relation* on a set X is a subset of $X \times X$.

$$R \subseteq \{(a, b) \mid a, b \in X\}$$

Example 1.2

$\leq$ is a relation on $\mathbb{R}$.

Example 1.3

$3 \leq \pi$ means that $(3, \pi) \in R$.

Example 1.4

$\{1\} \subseteq \{1, 2\}$ then $(\{1\}, \{1, 2\} \in R)$.

Some keywords:

- Reflexive. A relation R is reflexive if for any $a \in X$, $(a, a) \in R$.
- Transitive. If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

  Example 1.5

  $a \leq b \wedge b \leq c$ then $a \leq c$.

- Symmetric. If $(a, b) \in R$ then $(b, a) \in R$. (Doesn't hold often)
- Antisymmetric. If $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

Example 1.6

Let a set $V = \{1, 2, 3, 4\}$. We define the relation $<$ on the set V, thus we have the relation $E = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

We may put an edge from $1 \to 2$, $1 \to 3$, $1 \to 4$, $2 \to 3$, $\ldots$ so on. This becomes a graph. This relation is *antireflexive*, that is, $(a, a) \in R$ does *not* hold for any $a \in X$.

This relation is:

- Transitive. If $a < b$ and $b < c$ then $a < c$.
- Antisymmetric. The predicate never holds, hence the statement itself is vacuously true.

Definition. A *equivalence relation* is a transitive, reflexive, and symmetric relation.

Example 1.7

A dumb example is $=$. All the properties trivially hold. The relation $\equiv$ is also an equivalence relation.

Definition 1.8. A *partial order* is a transitive, reflexive, and antisymmetric relation.

Example 1.9

The relation $\leq$. It is transitive: $a \leq b$ and $b \leq c$ then $a \leq c$. It is also reflexive: $a \leq a$. It is also antisymmetric, since if $a \leq b$ and $b \leq a$, then $a = b$. The relation $\leq$ is also a *total order*.

Example 1.10

A directed acyclic graph (DAG) is a partial order. Verify some examples.

Definition 1.11. A *total order* is a partial order such that if $a, b \in X$, then either $(a, b) \in R$ or $(b, a) \in R$.

Example 1.12

The relation $\subseteq$ is a *partial order*.

It satisfies the three properties: $A \subseteq B$ and $B \subseteq C$ thus $A \subseteq C$ (transitivity); $A \subseteq A$ (reflexivity); if $A \subseteq B$ and $B \subseteq A$, then $A = B$ (antisymmetry).

But it's not a total order; we may take the counterexample $\{1, 2\}$ and $\{3, 4\}$.

Example 1.13

Let $X = \mathbb{Z}$, and $R = \{(a, b) \mid a \mid b\}$.

We have $(1, a) \in R$, $(2, 2a) \in R$, but also $(a, a) \in R$, so R is reflexive.

If $(a, b) \in R$ and $(b, c) \in R$, then $a \mid b$ and $b \mid c$.

Thus, $a \mid c$, so $(a, c) \in R$; hence R is transitive.

If $(a, b) \in R$, then $a \mid b$, but $b \nmid a$ for $(a, b) = (2, 4)$, so the relation R is *not* symmetric.

Moreover, it is not antisymmetric as well, because $a \mid b$ and $b \mid a$ does not necessarily imply $a = b$, since $a = -b$ also works.

But if $X = \mathbb{N}$, then the relation becomes antisymmetric, thus R is a partial order.

A relation can be both symmetric *and* antisymmetric! It's sort of like a clopen set, where the set is both closed and open.

Example 1.14

The equivalence relation $=$ is both symmetric and antisymmetric.

Also, a graph with every vertex having only a self-loop is both symmetric and antisymmetric. For a graph with n vertices, there are a total of 2^n such graphs. Each vertex can only either have a self-loop or not.

- How many reflexive relations are there on a directed graph with n vertices?
  There are $2^{2\binom{n}{2}} = 2^{n(n-1)}$ such graphs.
- How many symmetric relations are there on a directed graph with n vertices?
  There are $2^{\binom{n}{2}+n}$ such graphs.
- How many antisymmetric relations are there on a directed graph with n vertices?
  There are $3^{\binom{n}{2}} \cdot 2^n$ such graphs.
- How many reflexive *and* symmetric relations are there on a directed graph with n vertices?
  There are $2^{\binom{n}{2}}$ such graphs.
- What about transitivity? How many transitive relations are there on a directed graph with n vertices?

If $(a, b) \in R$, then with transitivity and symmetry, we have $(a, b) \in R \to (b, a) \in R \to (a, a) \in R$, but we don't know if $(a, b) \in R$ in the first place! That's why we need reflexivity.

Example 1.15

A directed graph with a lone vertex without an edge. It may be both transitive and symmetric, but never reflexive.

§2 Graphs

Definition 2.1. A *simple graph* does not have self-loops and has at most one edge between a pair of vertices.

Definition 2.2. We define $\deg v = |\{e \in E \mid v \in e\}|$.

Example 2.3

A self-loop counts as two degrees in both a directed and undirected graph.

Theorem 2.4 (Handshake lemma)

For all $G = (V, E)$, we have $\sum_{v \in V} \deg v = 2|E|$.

Specifically, for a directed graph, $\sum \text{indeg} v + \sum \text{outdeg} v = 2|E|$.

Definition 2.5. A *walk* is a series of edges $v_1v_2, v_2v_3, \ldots, v_lv_{l+1}$.

Definition 2.6. A *trail* is a walk with no repeated edges.

Definition 2.7. A *path* is a walk with no repeated vertices.

Definition 2.8. A *circuit* or a *closed trail* is a trail with $v_1 = v_{l+1}$.

Definition 2.9. A *cycle* is a circuit with no repeated vertices.

Definition 2.10. An *Eulerian trail* is a trail that uses every edge *exactly once*.

Theorem 2.11

A connected graph G has an Eulerian circuit iff all of the vertices of G have even degree.

Proof. For ($\Longrightarrow$), it is fairly easy to see that for any Eulerian circuit $C = v_1v_2, v_2v_3, \ldots, v_nv_1$, each vertex has even degree.

For ($\Longleftarrow$), we use induction. We induct on the number of edges, which we call n. The base case is $n = 0$, which is a point. (Or rather, start on $n = 3$, for which the graph must be C_3.)

Let $n > 3$, with $|E| = n$. Assume for any $G' = (V', E')$ with $|E'| < H$, we have $2 \mid \deg v$ for all $v \in V'$. Choose $v_1 \in V$, then take a maximal trail starting at v_1, which exists. A *maximal* trail means that such trail cannot be extended, which means for a trail $T = v_1v_2, \ldots, v_nv_{n+1}$, there does not exist $T' = v_1v_2, \ldots, v_nv_{n+1}, v_{n+1}v_{n+2}$.

For a trail $T = v_1v_2, v_2v_3, \ldots, v_lv_{l+1}$ for $l \in \mathbb{N}$, suppose for the sake of contradiction that $v_1 \neq v_{l+1}$. Then, we have $2 \nmid \deg v_1$, but $2 \mid \deg v$ for all $v \in V'$, which is a contradiction. So $v_1 = v_{l+1}$, which means T must be a circuit.

Now, take a circuit of largest possible size. If it contains all edges, it must include all vertices, hence it is Eulerian.

Suppose for the sake of contradiction that it did not contain some edge. Then, take the complement $\overline{T}$ of that circuit, which is connected and has the number of edges less than n.

Then, $\overline{T}$ also has an even degree on each of its vertices, thus it must also have an Eulerian circuit by the induction hypothesis. But then we can connect that circuit in $\overline{T}$ with the Eulerian circuit in T to make a larger Eulerian circuit, contradiction.

Hence, the largest Eulerian circuit has every edge in G, thus we are done. $\square$

Corollary 2.12

If G is connected, it has an Eulerian *trail* iff the number of vertices with odd degree is either 0 or 2.

Proof. If G is an Eulerian circuit, then each vertex has even degree.

If G is an Eulerian trail but not a circuit, then it starts and ends at different vertices, for which each of those start and end vertices have an odd degree. □

Example 2.13

The seven bridges of Königsburg does not have an Eulerian cycle.

Definition 2.14. A *subgraph* of $G = (V, E)$, denoted as $G' = (V', E')$, is a graph with $V' \subseteq V$ and $E' \subseteq E$.

§3 Planar graphs

Definition 3.1. A *planar graph* is a graph that can be drawn in a way that no edges cross each other.

Example 3.2

The complete graph on 5 vertices, denoted as K_5, cannot be drawn in any way that the edges do not cross, hence it is *nonplanar*.

Example 3.3

The complete bipartite graph $K_{3,3}$ is also nonplanar.

Example 3.4

The complete bipartite graph $K_{2,5}$ is planar. Actually, the complete bipartite graph $K_{2,n}$ is planar.

Example 3.5

The graph K_n where $2 \nmid n$ have an Eulerian circuit, since for each vertex we have even degree.

Theorem 3.6 (Euler's theorem)

For a connected, planar graph, we have $\chi = |V| - |E| + |F| = 2$, which is called the Euler characteristic of a graph.

The graph does not necessarily have to be simple (consider a self-loop, then compute the Euler characteristic χ).

	E	F	V
d = 3, n = 3	6	4	4
d = 4, n = 3	12	8	6
d = 5, n = 3	30	20	12
d = 3, n = 4	12	6	8
d = 3, n = 5	30	12	20

Table 1: A classification of all 5 platonic solids.

Definition 3.7. A *platonic solid* is a 3-dimensional object with all faces being regular polygons, with symmetry at every vertex, edge, and face. (One more constraint would be convexity.)

Theorem 3.8

There are only 5 platonic solids. Specifically, the only platonic solids are the cube, tetrahedron, octahedron, dodecahedron, and icosahedron, which all satisfy Euler's theorem.

Proof. All edges have 2 faces; all faces have n edges, where each face is a regular n-gon.

Moreover, all edges have 2 vertices; all vertices have d edges, where $d = 3, 4, 5$ for $n = 3$, $d = 3$ for $n = 4$, and $d = 3$ for $n = 5$. For $n \geq 6$, we need at least three faces for a vertex, but that's impossible since the sum of the interior degree is ≥ 360, which is absurd. Now, using the previous relations, we get $n|F| = 2|E|$ and $2|E| = d|V|$.

Hence, the result follows.

$\square$

Theorem 3.9

A platonic solid can be projected to form a planar graph.

Definition 3.10. A tree is a connected graph with no cycles.

Theorem 3.11

Removing an edge from a cycle never breaks connectedness.

Also, removing an edge does not make a planar graph nonplanar.

Definition 3.12. A face is a connected component in $\mathbb{R}^2 \setminus \bigcup_{e \in E} e$.

Theorem 3.13

Every edge has 1 or 2 faces.

Theorem 3.14

In a connected graph, every vertex has at least 1 edge.

Theorem 3.15

In a planar connected graph, if $|E| \geq 3$, all faces have at least 3 edges.

Theorem 3.16

In a planar connected graph, $3|F| \leq 2|E|$ and $|E| \leq 3|V| - 6$.

Proof. Double counting. □

Corollary 3.17

In a planar connected graph, the average degree of vertices is strictly less than 6.

Proof. We have $\sum_v \deg v = 2|E| \leq 6|V| - 12$, so $\overline{\deg v} < 6$. □

Definition 3.18. The *dual graph* of a planar graph G is a graph that has a vertex for each face of G. The dual graph has an edge for each pair of faces in G that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.

Example 3.19

The dual of a platonic solid is still a platonic solid.

Example 3.20

In fact, the dual of a tetrahedron is a tetrahedron; the dual of a cube is an octahedron; the dual of an icosahedron is a dodecahedron (and vice versa).

Exercise 3.21. Find the best bound on $|E|$, given that $|E| \geq g$ and there are no cycles of length $< g$.

Proof. If there is no cycle at all, then $|E| = |V| - 1$. Else, every cycle has at least g edges, hence $g|F| \leq 2|E|$, which means

$$|E| \leq \frac{g}{g-2}(|V| - 2)$$

□

Corollary 3.22

We have $\sum_v \deg v = 2|E| \leq \frac{2g}{g-2}(|V| - 2)$, so $\overline{\deg v} \leq \frac{2g}{g-2} - \frac{4g}{g-2}\frac{1}{|V|} < \frac{2g}{g-2}$.

Corollary 3.23

For bipartite graphs, we have a lower bound on $|E|$, hence we may actually apply this. (For example, for $K_{3,3}$, we have $|E| \geq 4$ and there is no cycle of length < 4.) Taking the contrapositive, if a graph with a lower bound on $|E|$ does not satisfy this inequality, then we may immediately realize that it is nonplanar.

For K_5, we have $|V| = 5$ and $|E| = 10$, so if we assume it is planar, $|E| \leq 3|V| - 6$, but then $10 \leq 9$ is false, hence K_5 is nonplanar.

Corollary 3.24

If a graph has K_5 or $K_{3,3}$ as a subgraph, it is nonplanar. Moreover, a graph with a topological embedding of K_5 or $K_{3,3}$ is also nonplanar. (If you get a K_5 or $K_{3,3}$ by contraction or deletion, then it is nonplanar.)

Definition 3.25. A topological embedding allows edge contraction (which creates a *minor*), vertex deletion, and edge deletion.

Theorem 3.26 (Kuratowski's theorem)

G is nonplanar if and only if it contains a topological embedding of K_5 or $K_{3,3}$.

Proof. https://math.uchicago.edu/~may/REU2017/REUPapers/Xu,Yifan.pdf □

Theorem 3.27

Every connected, simple, planar graph can be colored with at most 6 colors.

Proof. We induct on the number of vertices. The base case is obvious.

For any connected, planar, simple graph G, there exists a vertex v with degree ≤ 5, since $\overline{\deg v} < 6$, by the probabilistic method. Then, delete v, which gives a legal 6-coloring, and put back v, where we choose the color that's not chosen, hence we are done. □

Theorem 3.28

Every connected, simple, planar graph can be colored with at most 5 colors.

Proof. Suppose FTSOC that for every vertex, there were at least 5 edges, and its adjacent vertices were all colored differently. Consider the connected component only consisting of 1's and 3's (we may label the colors arbitrarily). If we delete v and get 1 and 3 to be not connected, then we are done; if not, then 2 and 4 are disconnected, since it is planar. □

§4 Ramsey theory

In Ramsey theory, we basically find a substructure that appears given some size of the graph.

Example 4.1

For K_6 or higher, we can always find a monochromatic K_3 subgraph.

Example 4.2

For K_5, we may construct a counterexample by hand. (All the diagonals in red, and all the edges in white)

Theorem 4.3

For any given positive integers r and k, there is some number N such that if the integers $\{1, 2, \ldots, N\}$ are colored, each with one of r different colors, then there are at least k integers in arithmetic progression whose elements are of the same color.

Theorem 4.4 (Ramsey's theorem)

For any $r, s \in \mathbb{N}$, $\exists N = R(r, s)$ such that if $K_{R(r,s)}$ has its edges colored in 2 colors, red and blue, it must contain either a blue K_r or a red K_s subgraph.

Basically, there exist monochromatic cliques in any edge labeling (with colors) of a sufficiently large complete graph.

Example 4.5

Find the largest square grid without any monochromatic squares.

Example 4.6

$R(3, 3) = 6$, $R(2, 4) = 4$.

We know that $R(2, s) = s$ and $R(r, 2) = r$.

Claim — $R(r, s) \leq R(r-1, s) + R(r, s-1)$.

Proof. Let $G = K_{R(r-1,s)+R(r,s-1)}$. We want to show that G contains a blue K_r or a red K_s. Take out a vertex v, then split the other vertices into whether they are connected with a red or blue edge: let them be $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Excluding v, we have $R(r-1, s) + R(r, s-1) - 1$ vertices. Then, by pigeonhole, $|V_1| \geq R(r-1, s)$ or $|V_2| \geq R(r, s-1)$. We only deal with the case $|V_1| \geq R(r-1, s)$, since the other case is totally symmetric.

For $|V_1| \geq R(r-1, s)$, if there was a blue K_s, then we're done. Otherwise, connected every vertex in K_{r-1} with v. The end. $\square$

Theorem 4.7 (Extended Ramsey's theorem)

For any $n_1, n_2, \ldots, n_c \in \mathbb{N}$, $R(n_1, n_2, \ldots, n_c)$ is the minimum number of vertices required to have a K_{n_i} subgraph of color i for at least one color i.

Claim — $R(n_1, n_2, \dots, n_c) \le R(n_1, n_2, \dots, n_{c-2}, R(n_{c-1}, n_c))$.

Proof. Either there is a K_{n_i} subgraph for $1 \le i \le c-2$, or there exists a $K_{R(n_{c-1}, n_c)}$ of combined colors $c-1$ and c.

If we have a $K_{R(n_{c-1}, n_c)}$ subgraph in two colors $c-1$ and c, then it has a subgraph of $K_{n_{c-1}}$ in $c-1$ or K_{n_c} in c. □

§5 Linear algebra

Definition 5.1. The trace $\mathrm{Tr}(A)$ of a square matrix A is the sum of the diagonals.

Example 5.2

$\mathrm{Tr}(I_n) = n$.

Theorem 5.3

For two square matrices A, B, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Corollary 5.4

The trace of cyclic permutations is the same.

Example 5.5

$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$.

However, $\mathrm{Tr}(ABC) \ne \mathrm{Tr}(ACB)$.

Exercise 5.6. Let J_n be a matrix with all entries being 1. We know that $\mathrm{Tr}(J_n) = n$. What is $(J_n)^2$? What is $(J_n)^k$?

Proof. $(J_n)^2 = nJ_n$, since for a connected graph K_n with self-loops we can move back to the first point at the penultimate step. Generally, $(J_n)^k = n^{k-1}J_n$. □

Definition 5.7. The adjacency matrix of a labeled graph is a matrix with rows and columns labeled by graph vertices, with the position (v_i, v_j) having the number of edges (v_i, v_j), counting with multiplicity. For (v_i, v_i), a self-loop is considered to add 1 to the adjacency matrix.

Remark. We only care about simple undirected graphs.

For an adjacency matrix A of a simple graph, $\mathrm{Tr}(A) = 0$. Moreover, A is symmetric.

Exercise 5.8. $\mathrm{Tr}(A^3) = 6 \times \#$ of triangles.

Exercise 5.9. $\mathrm{Tr}(JA) = \sum \deg(v_i) = 2|E|$.

Theorem 5.10

$(A^\ell)_{ij}$ is the number of walks from i to j of length ℓ.

Proof. We know $(BA)_{ij} = \sum_{k=1}^{n} B_{ik}A_{kj}$. Hence,

$$(A^\ell)_{ij} = (A^{\ell-1}A)_{ij} = \sum_{k=1}^{n} A_{ik}^{\ell-1}A_{kj}$$

at which point we are done by induction. □

Definition 5.11. Define $G_1 \oplus G_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2)$, which is an adjoining of A_1 and A_2, the adjacency matrices of G_1 and G_2 respectively.

Definition 5.12. Define $G_1 \vee G_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2 \sqcup \{(x, y) \mid x \in V_1,\ y \in V_2\})$, where we adjoin A_1 and A_2 and fill in the remaining elements with 1.

Definition 5.13. Define the complement of a graph $G = (V, E)$ to be $\overline{G} = (V, (V \times V) - E - \{(v, v) \mid v \in V\})$.

Corollary 5.14

$G + \overline{G} = K_n$.

Corollary 5.15

$A(G) + A(\overline{G}) = A(K_n) = J_n - I_n$, hence $A(\overline{G}) = J_n - I_n - A(G)$.

Theorem 5.16 (Ramsey's theorem)

For all graphs with $|V| \geq R(a, b)$, either G contains K_a or $\overline{G}$ contains K_b.

Example 5.17 ($R(3, 3)$)

For $R(3, 3)$, either G contains K_3 or $\overline{G}$ contains K_3, that is, either $\mathrm{Tr}(A(G)^3) > 0$ or $\mathrm{Tr}(A(\overline{G})^3) > 0$.

Proof. Assume $\mathrm{Tr}(A(G)^3) = 0$, then

$$
\begin{aligned}
&\mathrm{Tr}(A(\overline{G})^3)\\
&= \mathrm{Tr}((J - I - A(G))^3)\\
&= \mathrm{Tr}(J^3 - 3J^2 + 3J - I - A^3 + AAJ + AJA + JAA\\
&\quad - 3A^2 - AJJ - JAJ - JJA + 3AJ + 3JA - 3A)\\
&= \mathrm{Tr}(J^3) - 3\,\mathrm{Tr}(J^2) + 3\,\mathrm{Tr}(J) - \mathrm{Tr}(I_n) - \mathrm{Tr}(A^3)\\
&\quad + 3\,\mathrm{Tr}(A^2J) - 3\,\mathrm{Tr}(A^2) - 3\,\mathrm{Tr}(J^2A) + 6\,\mathrm{Tr}(JA) - 3\,\mathrm{Tr}(A)\\
&= n^3 - 3n^2 + 3n - n - \mathrm{Tr}(A^3)\\
&\quad + 3\,\mathrm{Tr}(A^2J) - 3\,\mathrm{Tr}(A^2) - 3n\,\mathrm{Tr}(JA) + 6\,\mathrm{Tr}(JA) - 3\,\mathrm{Tr}(A)\\
&= n^3 + O(n^2) - \mathrm{Tr}(A^3) + 3\,\mathrm{Tr}(A^2J) - 3\,\mathrm{Tr}(AJ^2)\\
&= n^3 + O(n^2) + 3\,\mathrm{Tr}(A^2J - AJ^2)\\
&= n^3 - 3\,\mathrm{Tr}\left(A\frac{JJ}{n}(J - A)\right) + O(n^2)\\
&= n^3 - 3\sum_{i=1}^{n} n\deg v_i - \deg {v_i}^2 + O(n^2)\\
&\geq n^3 - \frac{3n^3}{4} + O(n^2) = \frac{n^3}{4} + O(n^2)
\end{aligned}
$$

For sufficiently large n, we have $\mathrm{Tr}(A(\overline{G})^3) > 0$, hence we are done. □

§6 Inner product spaces

Definition 6.1. Define the *adjoint* $\mathbf{A}^*$ of the matrix $\mathbf{A}$ as $\overline{\mathbf{A}}^T$.

Definition 6.2. Define the inner product $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n} \overline{x_i} y_i = \overline{\mathbf{x}}^T \cdot \mathbf{y}$.

Lemma 6.3

$\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}^*\mathbf{y} \rangle$

Proof. $\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x})^*\mathbf{y} = \mathbf{x}^*(\mathbf{A}^*\mathbf{y}) = \langle \mathbf{x}, \mathbf{A}^*\mathbf{y} \rangle$. □

Consider a nice matrix $\mathbf{A}$ for which $\mathbf{A}^* = \mathbf{A}$. Then, consider the eigenvalue λ for which $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Because $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \langle \lambda\mathbf{x}, \mathbf{x} \rangle = \lambda\mathbf{x}^*\mathbf{x}$, we have $\lambda = \overline{\lambda}$, and thus $\lambda \in \mathbb{R}$.

If $\mathbf{v}_1, \lambda_1$ and $\mathbf{v}_2, \lambda_2$ are two different eigenvalues, that is, $\lambda_1 \neq \lambda_2$. Then, we may verify that $\langle \mathbf{A}\mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{A}\mathbf{v}_2 \rangle$, which implies $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$, so $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

Suppose $\dim(E_{\lambda_1}) = d$. Take an orthonormal basis in each eigenspace. Hence, there exists an orthonormal eigenbasis, $v_1, \ldots, v_n$ eigenvectors with $|v_i| = 1$ and $< v_i, v_j >= 0$.

§7 (n, d)-graphs

Let $|V| = n$, and $\deg(i) = d$ for all $i \in V$. Then, we call such a graph an (n, d)-graph.

Let $\mathbf{v} = \vec{1}$. Then, $\mathbf{A}\mathbf{v} = d\mathbf{v}$, hence $\mathbf{v}_1 = \vec{1}$ and $\lambda_1 = d$.

Theorem 7.1

For (n, d)-graphs, $\lambda_1 = d$ and $|\lambda_j| \leq d$ for all $1 \leq j \leq n$.

Proof. Suppose $\vec{x}$ is an eigenvector. Let $|x_i| = \max_{1 \leq j \leq n} |x_j|$. Then,

$$|(\mathbf{A}\vec{x})_i| = \left| \sum_{k=1}^{n} A_{ik} x_k \right| \leq \sum_{k=1}^{n} |A_{ik}||x_k| \leq \sum_{k=1}^{n} |A_{ik}||x_i| = |x_i| \cdot \sum_{k=1}^{n} A_{ik} = |x_i| \deg(i) = d\,|x_i|$$

and for an eigenvalue λ such that $\mathbf{A}\vec{x} = \lambda\vec{x}$, we have that $|\lambda\vec{x}_i| \leq d|x_i|$, thus $|\lambda| \leq d$. □

We order eigenvalues as follows: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$.

Theorem 7.2

$|\{\lambda_i \mid \lambda_i = d\}| = \#$ of connected components of G.

Proof. Let C be a connected component, whence $C \subseteq V$. Let $\mathbf{v} = \sum_{i \in C} \mathbf{e}_i$, where $\mathbf{e}_i$ is the canonical basis (i.e., $\mathbf{e}_{ii} = 1$ and $\mathbf{e}_{ij} = 0$ for $j \neq i$).

Consider $(\mathbf{A}\mathbf{v})_j = (\mathbf{A} \sum_{i \in C} \mathbf{e}_i)_j = \sum_{i \in C} (\mathbf{A}\mathbf{e}_i)_j = \sum_{i \in C} \mathbf{A}_{ji}$.

If $j \notin C$, then $\mathbf{A}_{ji} = 0$ for all $i \in C$, so $(\mathbf{A}\mathbf{v})_j = 0$, hence $(\sum_{i \in C} \mathbf{e}_i)_j = 0$.

If $j \in C$, then $\deg(j) = d$, so j has d vertices in C, where it is connected to $\sum_{i \in C} \mathbf{A}_{ji} = d$, so $\mathbf{v}_j = \left(\sum_{i \in C} \mathbf{e}_i\right)_j = \vec{1}$. Hence, $\mathbf{A}\mathbf{v} = d\mathbf{v}$ in this case. □

Proof. Suppose x is an eigenvector of eigenvalue d. Consider

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}(x_i - x_j)^2 \\
&= \sum_i \left(\sum_j A_{ij} \right) x_i^2 + \sum_j \left(\sum_i A_{ij} \right) x_j^2 - 2 \sum_i \sum_j A_{ij} x_i x_j \\
&= \sum_i d x_i^2 + \sum_j d x_j^2 - 2 \sum_j \sum_i (x_i A_{ij})\, x_j \\
&= 2d\,\|\mathbf{x}\|^2 - 2\mathbf{x}^T \mathbf{A}\mathbf{x} \\
&= 2d\|\mathbf{x}\|^2 - 2d\mathbf{x}^T \mathbf{x} \\
&= 2d\|\mathbf{x}\|^2 - 2d\|\mathbf{x}\|^2 = 0
\end{aligned}$$

Now, because $A_{ij}(x_i - x_j)^2 \geq 0$, we have $A_{ij}(x_i - x_j)^2 = 0\ \forall i, j$.

Hence, $x_i = x_j$ whenever $ij \in E$, that is, $x_i = x_j\ \forall i, j \in C$.

Note that $\mathbf{x} = \sum_{C_j} k\mathbf{v}_j$ where $\mathbf{v}_j = \sum_{i \in C_j} \mathbf{e}_i$. Thus, the number of eigenvectors with $\lambda_i = d$ is less than or equal to the number of connected components. □

Corollary 7.3

We have that $\lambda_2 = d \iff G$ is disconnected.

Theorem 7.4

If G is bipartite, then $\lambda_n = -d$.

Proof. Take the vertex sets V_1 and V_2 such that $V_1 \cup V_2 = G$. Moreover, let E such that if $i, j \in V_1$ then $ij \notin E$ and if $i, j \in V_2$ then $ij \notin E$. Let $\mathbf{v} = \sum_{i \in V_1} \mathbf{e}_i - \sum_{j \in V_2} \mathbf{e}_j$, which has entry 1 if it is in the first set and -1 if it is in the second set. If $\ell \in V_1$, then $(\mathbf{Av})_\ell = \sum_{i=1}^n A_{\ell i} v_i = \sum_{i \in V_2} A_{\ell i} v_i = \sum_{i \in V_2} -A_{\ell i} = -d$. Similarly, if $\ell \in V_2$, then $\sum_{i=1}^n A_{\ell i} v_i = d$, hence we get an eigenvalue $-d$. $\square$

Theorem 7.5

If $\lambda_n = -d$ and G is connected, then G is bipartite.

Proof. We already know that we can check whether G is connected by looking at λ_2. Consider $\sum\sum A_{ij}(x_i + x_j)^2$. If $\mathbf{x}$ has $\lambda = -d$, then $\sum\sum A_{ij}(x_i + x_j)^2 = 2d\|\mathbf{x}\|^2 + 2\sum A_{ij} x_i x_j = 2d\|\mathbf{x}\|^2 + 2\mathbf{x}^T\mathbf{Ax} = 2d\|\mathbf{x}\|^2 - 2d\|\mathbf{x}\|^2 = 0$. Hence, $A_{ij}(x_i + x_j)^2 = 0$, so $x_i = -x_j$ for all $ij \in E$.

Let $R = \{i \mid x_i = 0$, $S = \{i \mid x_i > 0$, and $T = \{i \mid x_i < 0\}$. But if $i \in R$, then $x_i = 0$, so either $R = V$ or $R = \emptyset$, but since $\vec{0}$ cannot be an eigenvector, we must have $R = \emptyset$. If $i \in S$, then $x_i > 0$, so for any $ij \in E$, $x_j < 0$, where $j \in T$. Similarly, if $i \in T$, then $j \in S$. Hence, G is bipartite. $\square$

If there exists a homomorphism $\varphi : G \to \{\pm 1\}$ with $\varphi(S) = \{-1\}$, then Γ is bipartite.

Proof. Suppose $\exists \varphi : G \to \{\pm 1\}$ and $\phi(S) = -1$. Let $A = \varphi^{-1}(\{1\})$ and $B = \varphi^{-1}(\{-1\})$. Suppose $\{g_1, g_2\} \in E$. Then, $\exists s \in S$ such that $sg_1 = g_2$. Hence, $\varphi(s)\varphi(g_1) = \varphi(sg_1) = \varphi(g_2)$, which implies $-\varphi(g_1) = \varphi(g_2)$, thus Γ is bipartite. $\square$

If Γ is bipartite and connected, then there exists a homomorphism $\varphi : G \to \{\pm 1\}$ with $\varphi(S) = \{-1\}$.

Proof. Suppose Γ is bipartite and connected. Let A, B be the two sets of V with $A \cup B = V$ such that $\{g_1, g_2\} \in E$ implies either $g_1 \in A \wedge g_2 \in B$ or vice versa. Note that $E(A, A) = E(B, B) = \emptyset$. If $e \in A$, then $\varphi(A) = \{1\}$ and $\varphi(B) = \{-1\}$. We know that $\varphi(e) = 1$. It suffices to show that $\varphi(gh) = \varphi(g)\varphi(h)$.

Observe that $\{e, s\} \in E$ for any $s \in S$. Hence, $s \in B$, and thus $\varphi(S) = \{-1\}$. Suppose $g \in A$ and $h \in A$. Then, $\exists s_1, s_2, \ldots, s_k$ connecting e to g, and $t_1, \ldots, t_\ell$ connecting e to h. Hence, the path from e to hg can be constructed as follows: $1, s_1, s_2 s_1, \ldots, g, t_1 g, t_2 t_1 g, \ldots, t_{\ell-1} \ldots t_1 g, hg$. Therefore, the parity of the length is preserved $(k + \ell)$, and thus $hg \in A$, which implies $\phi(hg) = 1 = \phi(h)\phi(g)$. $\square$

Definition 7.6. The *spectrum* is the set of eigenvalues.

A graph is disconnected iff $\lambda_2 = d$. Equivalently, a graph is connected iff $\lambda_1 - \lambda_2 > 0$. We call $\lambda_k - \lambda_{k+1}$ the *spectral gap*.

Definition 7.7. For $S \subseteq V$, we have $\overline{S} = V \setminus S$. Define the edges between two sets S and T be $E(S, T) := \{\{s, t\} \in E \mid s \in S,\ t \in T\}$. We denote the *boundary* of a vertex set S to be $\partial S := E(S, \overline{S})$.

Definition 7.8. The partition of V into S and $\overline{S}$ is called a *cut*. An edge $e \in \partial S$ is said to *cross the cut*. We define $|\partial S|$ to be the *size* of the cut.

Definition 7.9. Define the *expansion ratio* $h(G) := \min_{1 \leq |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}$.

Example 7.10

For C_n, we have $h(G) = \dfrac{2}{\lfloor \frac{n}{2} \rfloor}$.

Lemma 7.11

$h(G) > 0 \iff G$ is connected.

Notice that G is connected $\iff \lambda_1 - \lambda_2 > 0$. Hence, let us think of a way to connect $h(G)$ and $\lambda_1 - \lambda_2$.

Theorem 7.12 (Cheager's inequality)

For (n, d)-graph G, let $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then, we have

$$\frac{\lambda_1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2\lambda_1(\lambda_1 - \lambda_2)}$$

Example 7.13

Consider C_n, for which we have $d = \lambda_1 = 2$. Then,

$$h(C_n) = \frac{2}{\lfloor \frac{n}{2} \rfloor} < \frac{2}{n/2 - 1} = \frac{4}{n-2}$$

which implies $2 - \lambda_2 < \frac{8}{n-2}$ by Cheager's inequality, hence $2 > \lambda_2 > 2 - \frac{8}{n-2}$.

Example 7.14

Consider K_n, for which $d = \lambda_1 = n - 1$. The adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \ddots & & & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}$$

satisfies $A = J - I$. For $\det(A - \lambda I) = 0$, we may take $\lambda = -1$, for which $\det(A - \lambda I) = \det(J) = 0$. Hence, $\dim(E_\lambda) = n - \operatorname{rank}(A - \lambda I) \operatorname{rank}(J) = 1$ by rank-nullity. Thus, $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$.

Example 7.15

For $S \subseteq V$, we have $|S| = s$ and thus $|\partial S| = s \cdot (n - s)$. Hence, $h(K_n) = \min_{1 \le |S| \le \frac{n}{2}} \frac{|\partial S|}{|S|} \left\lceil \frac{n}{2} \right\rceil$.

Definition 7.16. Let G be a finite group, with $|G| = n$. Then, $S \subseteq G$ is symmetric if S is closed under inversion, that is, $s \in S \implies s^{-1} \in S$. The *Cayley graph* $\Gamma(G, S)$ is a graph with $V = G$ and $E = \{\{g, sg\} \mid g \in G,\ s \in S\}$.

Note that $\Gamma(G, S)$ is $|S|$-regular.

Example 7.17

Consider $\Gamma(\mathbb{Z}_6, \{\pm 1, \pm 5\})$.

Definition 7.18. We say S *generates* G if $\forall g \in G$, $\exists s_1, \dots, s_k \in S$ such that $g = s_1 s_2 \dots s_k$.

Lemma 7.19

Γ is connected $\iff$ S generates all of G.

Proof. Proof of ($\implies$). Suppose Γ is connected. Then, $\forall g \in G$, there exists a path from $e \in G$ to g, namely $(e, a_1), (a_1, a_2), \dots, (a_{n-1}, g) \in E$. Let $e = a_0$ and $g = a_n$. We know that $\exists s_k \in S$ such that $s_k a_{k-1} = a_k$. Hence, $g = a_n = s_n a_{n-1} = \cdots = s_n s_{n-1} \dots s_1 e$. ■

Proof of ($\impliedby$). Assume S generates all of G. Let $g, h \in G$. Then, $hg^{-1} \in G$, so $\exists s_1, s_2, \dots, s_n \in S$ such that $hg^{-1} = s_n \dots s_1$. Hence, $h = s_n s_{n-1} \dots s_1 g$. Hence, there exists a path from g to h, namely, $(g, s_1 g), (s_1 g, s_2 s_1 g), \dots, (s_{n-1} \dots s_1 g, s_n \dots s_1 g = h)$. ■

□

Let ω_1 be an n^{th} root of unity and $\omega_k = \omega_1^k$.

For the eigenvalues λ_k, we have

$$\lambda_k = \sum_{i \in S} \omega_k^i$$

(note that λ's are not necessarily in decreasing order).

Consider $C_n = \Gamma(\mathbb{Z}_n, \{\pm 1\})$, for which

$$\lambda_k = \sum_{i \in \{\pm 1\}} \omega_k^i = \omega_k + \omega_k^{-1} = 2\Re(\omega_k) = 2\cos\left(\frac{2\pi k}{n}\right)$$

We often consider the *rescaled spectral graph*, where the eigenvalues are divided by $\max_i \lambda_i$, in order to compare different graphs more easily.

Problem 7.20. Which ones of $S = \{\pm 1, \pm k\}$ are the most interconnected, and which ones are the least interconnected?

§8 Graph maps

Let $S \subseteq G$, symmetric, and $\varphi : G \to H$ is a group homomorphism. There is a graph map $(\psi_v, \psi_e) : \Gamma(G,S) \to \Gamma(H,T)$ so long as $\varphi(S) \subseteq T$.

Let $\Gamma_1 = \Gamma(G,S)$, $V_1 = G$, $\Gamma_2 = \Gamma(H,T)$, $V_2 = H$.

Let $\psi_v = \varphi$ and $\psi_e = \varphi \times \varphi$, where $\psi_e(g, sg) = (\varphi(g), \varphi(sg)) = (\varphi(g), \varphi(s) \cdot \varphi(g))$ for $g \in G$ and $s \in S$. Note that $\varphi(S) \subseteq T$, thus $\varphi(s) \in T$.

Take a Cayley graph $\Gamma = \Gamma(G,S)$. Then, $\forall g, h \in G$, there exists an automorphism graph map $\psi = (\psi_v, \psi_e)$ such that $\psi_v(g) = h$. Here, $\varphi : G \to G$ is as follows: $\phi : x \mapsto xg^{-1}h$, so $\varphi(g) = gg^{-1}h = h$.

> **Remark 8.1.** Note that this is **not** a homomorphism, since $\varphi(e) = g^{-1}h$.

We let $\psi_v = \varphi$ and $\psi_e(x, sx) = (\varphi(x), s\varphi(x)) \in E(\Gamma)$. Moreover, note that we have $\varphi(sx) = s\varphi(x)$, since $\varphi(sx) = sxg^{-1}h = s\varphi(x)$. Then, φ is bijective, since we have inverses $\psi_e(\varphi^{-1}(x), s\varphi^{-1}(x)) = (x, sx)$.

Exercise 8.2. Prove that all Cayley graphs are vertex-transitive.

Let $G = \mathbb{Z}_n$ and $H = \left\{ e^{\frac{2\pi i k}{n}} \mid k \in \{0, 1, \ldots, n-1\} \right\}$. There is a natural isomorphism φ between G and H, where $\varphi : G \to H$ with $\varphi : x \mapsto e^{\frac{2\pi i x}{n}}$.

For a finite group G, we have a map $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ tied to a representation $\mathrm{GL}_n(\mathbb{C})$. For the previous example, we have $\mathrm{GL}_1(\mathbb{C})$, that is, a matrix of a single complex number, which is just a single complex number.

There is a trivial representation $\varphi : G \to \{1\}$.

Let $\mathbf{v} = [\omega_k;\ \omega_k^2;\ \ldots;\ \omega_k^n]$. Then, $(\mathbf{A}\mathbf{v})_i = \sum_{j=1}^n a_{ij} v_j = \sum_{i=1}^n a_{ij} \omega_k^j$.

Note that $a_{ij} = 1$ iff $i - j \in S$. Hence, $(\mathbf{A}\mathbf{v})_i = \sum_{i=1}^n a_{ij} \omega_k^j = \sum_{j-i \in S} \omega_k^j = \sum_{j \in S} \omega_k^{j+i} = \omega_k^i \sum_{j \in S} \omega_k^j$. Moreover, $v_i = \omega_k^i$. This means that $\mathbf{A}\mathbf{v} = \left(\sum_{j \in S} \omega_k^j \right) \mathbf{v}$, whence $\left(\sum_{j \in S} \omega_k^j \right)$ is an eigenvalue, that is, $\mu_k = \sum_{j \in S} \omega_k^j$.

We have $\mu_0 = \sum_{j \in S} \omega_0^j = |S| = d$, which is the largest eigenvalue.

Let $S = \mathbb{Z}_n \setminus \{0\}$, that is, no self-loops. Moreover, $\mu_k = \sum_{j=1}^{n-1} \omega_k^j = \frac{1 - \omega_k^n}{1 - \omega_k} - 1 = -1$, provided $\omega_k = 1$, i.e., $k \neq 0$.

Then, for $\lambda_1 \geq \lambda_2 \geq \ldots$, we have $\lambda_1 - \lambda_2 = n$, so the spectral gap $\frac{\lambda_1 - \lambda_2}{\lambda_1'} = \frac{n}{n-1} = 1 + \frac{1}{n-1} \to 1$, and its maximum value is 2.

Consider the cyclic graph C_n, where $S = \{\pm 1\}$. Then, $\mu_k = \sum_{j \in S} \omega_k^j = \omega_k + \omega_k^{-1} = 2\Re(\omega_k) = 2\cos(2\pi k/n)$. So, $\mu_1 = 2\cos(2\pi/n)$ is the second-largest eigenvalue after $\mu_0 = 1$. Thus, the rescaled spectral gap is $\frac{\lambda_1 - \lambda_2}{\lambda_1} = \frac{\mu_0 - \mu_1}{\mu_0} = 1 - \cos(2\pi/n) \to 0$, so the cyclic graphs are the most disconnectable graphs.

Consider another family of examples: $S = \{\pm 1, \pm k\}$, where $k \neq \frac{n}{2}$. Then, $\mu_j = \omega_j + \omega_j^{-1} = \omega_j^k, \omega_j^{-k}$.

Which of these is the least connected (i.e., most disconnectable)? We need to find the largest λ_2. That is, we need to compute $\max_{k \neq 0, \frac{n}{2}, \pm 1} \max_{j \neq 0} \mu_j = \max_{k \neq 0, \frac{n}{2}, \pm 1} \max_{j \neq 0} \omega_j + \overline{\omega_j} + \omega_j^k + \overline{\omega_j^k} = 2\Re(\omega^j) + 2\Re(\omega^{jk})$. We want to find λ_2, so the maximum value we can get should be the case where $k = \frac{n}{j}$ and $j \neq 1, 2$, which gives the value of $\omega_3 + \overline{\omega_3} + 1 + 1 = 2(1 + \cos(6\pi/n))$. Thus, the highest real value is achieved when $j = 3$ and $k = \frac{n}{3}$, given that $3 \mid n$. Otherwise, it is the best to take $j = 1$ and $k = 2$.

> **Remark.** We hope that $j = 1$ and $k = 2$ is always better than $j = 3$ and $k = \frac{n}{3}$, because then we have to divide cases.

Looking at $j = 1$ and $k = 2$, we have $\omega + \overline{\omega} + \omega^2 + \overline{\omega^2} = 2(\cos(2\pi/n) + \cos(4\pi/n))$. It remains to show that $f(x) = \cos(2\pi/x) + \cos(4\pi/x) - 1 - \cos 6\pi/x \geq 0$. Note that $f(x) = 2\sin^2(2\pi/x)(2\cos(2\pi/x) - 1)$, so the zeros of $f(x)$ are $x = \frac{2}{k}, \frac{6}{6k\pm1}$. We have $f(6) = 0$, $f(8) = \sqrt{2}/2 - 1 + \sqrt{2}/2 = \sqrt{2} - 1 > 0$. So, for all $x \geq 6$, we have $f(x) \geq 0$, which means $j = 1$ and $k = 2$ is always better for all $n \geq 6$. For $n < 6$, we may check manually, but there aren't many $n < 6$ with $3 \mid n$.

To find the most connected, we need to compute $\min_{k\neq 0, \frac{n}{2}, \pm1} \max_{j\neq 0} \mu_j$, which is much harder (minimax is hard in general).

Conjecture 8.3. If $2 \mid n$, $S = \{\pm 1, \frac{n}{2} \pm 1\}$ is the most interconnected.

$\mu_j = \omega_j + \overline{\omega_j} + \omega_j^{\frac{n}{2}-1} + \omega_j^{\frac{n}{2}+1}$. Find the max of μ_j, that is, find λ_2.

§9 Random walks on graphs

On (n, d, α)-graphs, where $\lambda = \max_{i\neq 1} |\lambda_i|$ and $\alpha = \frac{\lambda}{d}$. Note that $0 \leq \alpha \leq 1$.

What does it mean to have $\alpha < 1$?

> **Remark.** A graph is connected iff $\lambda_2 \neq d$, and it is not bipartite iff $\lambda_n \neq -d$.

Hence, such a graph is connected and not bipartite.

Definition 9.1. A *probability distribution* on V is a vector $\vec{p} = (p_1; p_2; \ldots; p_n)$ with $p_i \geq 0$ and $\sum p_i = 1$, meaning that we are at vertex i with probability p_i.

Definition 9.2. The *diffusion matrix* (normalized adjacency matrix) $\hat{A}$ is defined to be $\hat{A} := AD^{-1}$, where

$$D = \begin{bmatrix} \deg(v_1) & 0 & 0 & \ldots & 0 \\ 0 & \deg(v_2) & 0 & \ldots & 0 \\ \ddots & & & & \\ 0 & 0 & 0 & \ldots & \deg(v_n) \end{bmatrix} = dI$$

For a regular graph, $D^{-1} = \frac{1}{d}I$, and thus $\hat{A} = \frac{1}{d}A$.

For a probability distribution $\vec{u} = \frac{\vec{1}}{n}$, we have $\hat{A}\vec{u} = \frac{1}{d}A\frac{\vec{1}}{n} = \frac{1}{dn}A\vec{1} = \frac{1}{dn}d\vec{1} = \frac{1}{n}\vec{1} = \vec{u}$. Thus, $\vec{u}$ is an eigenvector satisfying $\hat{A}\vec{u} = \vec{u}$.

Theorem 9.3

Given $G = (V, E)$ with $\vec{p} = (p_1; p_2; \ldots; p_n)$ being the initial probability distribution, then $\hat{A}^k\vec{p}$ is the distribution after k time steps of random movement.

Proof. Arriving at i after $k+1$ steps requires to arrive at j after k steps, then move from j to i. This means that

$$(\vec{p}_{k+1})_i = \sum_j \hat{A}_{ij}(\vec{p}_k)_j = (\hat{A}\vec{p}_k)_i$$

which implies $\vec{p}_{k+1} = \hat{A}\vec{p}_k$, hence by induction, $\vec{p}_k = \hat{A}^k\vec{p}_0 = \hat{A}^k\vec{p}$. □

Intuitively, the probability distribution should approach a uniform distribution. However, bipartite graphs and disconnected graphs have a problem: the probability oscillates back and forth. Hence, we only consider $\alpha < 1$.

Theorem 9.4

Let G be (n, d, α) with $\alpha < 1$, and $\vec{p}$ be any probability distribution. Then,

$$\hat{A}^k\vec{p} \to \vec{u} = \frac{\vec{1}}{n}$$

Proof. It suffices to show that $|\hat{A}^k\vec{p} - \vec{u}| \to 0$. But note that $\vec{p}_k = \hat{A}^k\vec{p}$, so

$$|\vec{p}_k - \vec{u}| = |\hat{A}\vec{p}_{k-1} - \vec{u}| = |\hat{A}(\vec{p}_{k-1} - \vec{u})|$$

Then, we have $\vec{p}_{k-1} - \vec{u} = \vec{q} = \sum_{i=1}^r q_i v_i$, where $q_i \in \mathbb{R}$ and v_i are the orthonormal eigenbasis of A (not rescaled). When speaking of λ_i, we refer to the eigenvalues of the original matrix A.

Thus, $|\hat{A}(\vec{p}_{k-1} - \vec{u})| = |\hat{A}\vec{q}| = |\sum q_i \hat{A} v_i| = |\sum \frac{q_i A v_i}{d}| = \frac{1}{d}|\sum q_i \lambda_i v_i| = \frac{1}{d}\sqrt{\sum (q_i\lambda_i)^2}$. Now, notice that $\frac{1}{d}\sqrt{q_1^2\lambda_1^2 + \sum_{i=2}^n q_i^2\lambda_i^2} \le \frac{1}{d}\sqrt{q_1^2 d^2 + \sum_{i=2}^n q_i^2\lambda^2} = \sqrt{q_1^2 + \sum_{i=2}^n q_i^2\alpha^2} \le \sqrt{q_1^2 + \sum_{i=1}^n q_i^2\alpha^2} = \sqrt{q_1^2 + \alpha^2|q|^2} = \alpha|q|$, since $q_1 = \vec{q}\cdot\vec{v}_1 = (\vec{p}_{k-1} - \vec{u})\cdot\frac{\vec{1}}{\sqrt{n}}$, since we take an orthonormal basis, and thus $q_1 = \frac{\vec{p}_{k-1}\cdot\vec{1}}{\sqrt{n}} - \frac{\vec{u}\cdot\vec{1}}{\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{\vec{1}\cdot\vec{1}}{n\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}} = 0$. Therefore, $|\vec{p}_k - \vec{u}| = |\hat{A}(\vec{p}_{k-1} - \vec{u})| \le \alpha|\vec{p}_{k-1} - \vec{u}|$, so by induction, $|\vec{p}_k - \vec{u}| \le \alpha^k|\vec{p} - \vec{u}|$, and because $\alpha < 1$, we have $|\vec{p}_k - \vec{u}| \to 0$ as $k \to \infty$, exponentially. ☐

Exercise 9.5. Prove that staying in a bounded region has probability zero.

Problem 9.6. Let $p(x)$ be the probability of hitting M before 0, starting at x. Then, we have the recurrence $p(x) = \frac{1}{2}p(x+1) + \frac{1}{2}p(x-1)$, $p(0) = 0$, and $p(M) = 1$. Take the characteristic polynomial of the recurrence, getting $n^2 = 2n - 1$, so $n = 1$ being the double root, we have the roots 1^x and $x \cdot 1^x$, so $p(x) = a \cdot 1^x + b \cdot x \cdot 1^x = a + bx$, thus solving the system, we get $p(x) = \frac{x}{M}$.

Rather, we may regard this as a sequence of R's and L's, so in order to return after $2n$ steps, the probability is $\frac{\binom{2n}{n}}{2^{2n}}$, so the expected number of returns is

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}}$$

Let p be the probability of eventually returning to 0. The probability of returning n times is $p^n(1-p)$, so

$$\sum_{n=1}^{\infty} np^n(1-p) = \frac{p}{(1-p)^2}(1-p) = \frac{p}{1-p}$$

Then,

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} = \frac{p}{1-p}$$

and since the LHS diverges, $p \to 1$.

Using Stirling's approximation, we have

$$n! \sim n^n e^{-n}\sqrt{2\pi n}$$

so we perform a limit comparison test. Then,

$$\frac{\frac{(2n)!}{(n!)^2}}{2^{2n}} \sim \frac{\sqrt{4\pi n}}{2\pi n} = \frac{1}{\sqrt{\pi n}}$$

and because $\sum \frac{1}{\sqrt{\pi n}}$ diverges by the p-test, which means the LHS diverges, so $p \to 1$.

Exercise 9.7. Compute the probability of returning to 0 in $\mathbb{Z}^2$ and $\mathbb{Z}^3$.

In $\mathbb{Z}^2$, we have $E = \sum_{n=1}^{\infty} p_2(2n)$, where $p_n(k)$ is the probability of returning at least once in k moves in n dimensions, that is,

$$\begin{aligned} p_2(2n) &= \frac{\sum_{i=0}^{n} \frac{(2n)!}{(i!)^2((n-i)!)^2}}{4^{2n}} \\ &= \frac{\sum_{i=0}^{n} \frac{(2n)!}{n!n!} \cdot \frac{n^2}{(i!)^2((n-i)!)^2}}{4^{2n}} \\ &= \frac{\sum_{i=0}^{n} \binom{2n}{n}\binom{n}{i}^2}{4^{2n}} \\ &= \frac{\binom{2n}{n}^2}{4^{2n}} \sim \frac{1}{\pi n} \end{aligned}$$

thus $E \to \infty$, so we return at least once.

In $\mathbb{Z}^3$, we have $p_3(2n) = \frac{1}{6^{2n}} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(2n)!}{(i!)^2(j!)^2((n-i-j)!)^2} = \frac{1}{6^{2n}} \sum_{i=0}^{n} \binom{2n}{n}\binom{n}{i}^2\binom{2i}{i}$, which isn't really nice. But we can be handwavy, getting $p_3(2n) = \sum_{i=0}^{n} p_1(2i) \cdot p_2(2n - 2i) \cdot \mathbb{P}(\text{having } 2i) \sim \frac{3\sqrt{3}}{2} \cdot \frac{1}{(\pi n)^{\frac{3}{2}}}$, since the vast majority of cases reside in $i \approx \frac{n}{3}$. Thus, $E < \infty$, which means that we typically do not return to the origin.

§10 Graphs on manifolds

We may embed an arbitrary graph locally onto a sphere, which makes actual sense, since we can then think of "outer faces", which we previously considered as an imaginary face.

But what if we embed a graph on a torus? We know that for planar graphs, $V - E + F = 2$. Normally, we would get $V - E + F = 2$, but if we embed a graph on the inner loop of a torus, then we have $V - E + F = 1$, which is weird. We have that $\chi = 0$ for a torus, so $V - E + F = 0$ for a graph that uses both loops of a torus. In general,

Exercise 10.1. Which K_n and $K_{n,m}$ are not planar on a torus (i.e., toroidal) that satisfies $V - E + F \geq 0$?

Proof. For K_n, we have $|V| = n$, $|E| = \binom{n}{2}$, and $g = 3$, so $n \leq 7$. For $K_{n,m}$, we have $|V| = m + n$, $|E| = mn$, and $g = 4$, so $(m-2)(n-2) \leq 4$, thus $m = 1, 2$, $n = 1, 2$, $m = 3$, $n \leq 6$, $n = 3$, $m \leq 6$, $m = 4$, $n \leq 4$, $n = 4$, $m \leq 4$, $m = 5$, $n \leq 3$, $m = 6$, $n \leq 3$, and $n = 6$, $m \leq 3$.

For the actual construction. . . . □

Exercise 10.2. Do the same thing on the torus, a Möbius strip, a Klein bottle, and a projective plane.