Modal Logic

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This is a note on a series of lectures on modal logic, instructed by Pico Gilman, at the 2023 Ross Mathematics Program at Otterbein College.

§1 Propositional Logic

Definition 1.1. P, Q, R are statements (have a truth value), and $\neg, \wedge$ are the only operations. ($P \to Q$ is $\neg P \vee Q$)

§2 Modal Logic

Definition 2.1. Modal logic is a propositional logic with $\Box, \Diamond$.

- $\neg\Box\neg(P) = \Diamond(P)$.
- $\neg\Diamond\neg(P) = \Box(P)$.
- $(\neg\neg)\Diamond(\neg\neg) = \neg\Box\neg = \Diamond$.

Axioms:

- K (distribution axiom) $(\Box(P \to Q)) \to (\Box P \to \Box Q)$
- N (necessitation rule) $P \to \Box P$

§3 Temporal Modal Logic

Definition 3.1.
- $\Box =$ "is always true"
- $\neg\Box\neg = \Diamond =$ "is sometimes true"

> **Remark.** Temporal Modal Logic *cannot* satisfy the necessitation rule.

§4 Deontic Modal Logic

Definition 4.1. $\Box\phi =$ "ϕ is necessary"

§5 Topological Modal Logic

Definition 5.1. $\mathrm{int}(Y) = \{x \mid \exists U \ni x, U \subseteq Y\} = \{x \mid \exists U \ni x, \inf_U(\mathbb{1}_Y) = 1\}$.

Definition 5.2. X is a set; possible propositions $= 2^X$. τ is a topology on X.
$\Box(Y) = \mathrm{int}(Y)$.
$\Box P \to \Box\Box P$.
$\Box(Y) \vee \Box(Z) = \text{“}1\text{''} \to [\Box(Y \wedge Z) \neq \text{“}0\text{''}] \vee [Y = \text{“}0\text{''} \vee Z = \text{“}0\text{''}]$ would be the definition of connectedness.

§6 Poset-Topological Modal Logic

Let L be a poset (X, τ) be a topological space. Our proposition is L^x.

$A \vee B(x) = \sup(A(x), B(x))$. $\neg A(x) = \gamma(x)$. (We reverse the poset, by basically reversing the order of everything) $A \vee \neg A(x) = \max(L)$. $\Box A(x) = \sup_{U \supseteq x}(\inf_{u \in U} A(u))$.

Example 6.1 (Cofinite lattice)
Finite and cofinite lattices (union of finite and cofinite gives cofinite, intersection of finite and cofinite gives finite, ...) $\{x \subseteq \mathbb{N} \mid |x| < \infty \vee |\mathbb{N} \setminus x| < \infty\}$

Definition 6.2. An antichain is a subset of elements where no two distinct elements are not comparable.

Definition 6.3. For a poset L and $X \subseteq L$, define

$$S(X) := \{y \in L \mid \exists x \in X, y \leq x\}$$

Exercise 6.4. Prove that $S(S(X)) = S(X)$.

Exercise 6.5. Prove that $S(X) \cup S(X') = S(X \cup X')$.

Theorem 6.6
The following three statements are equivalent:

1. Let $X \subseteq L$, then $\exists Y \subseteq X$ s.t. $|Y| < \infty$ and $S(X) = S(Y)$.
2. Let $X \subseteq L$, then $Y \subseteq L$ s.t. $|Y| < \infty$ and $S(X) = S(Y)$.
3. L has no infinite ascending chains and infinite antichains.

Proof. *(3) $\implies$ (2).* For $X \subseteq L$, let $x_0 \in X$. Consider the chain $x_L > x_{L-1} > \cdots > x_0$, where the chain terminates at x_L, since there is no infinite ascending chain. Let $X' = X \setminus S(\{x_L\})$, then we have $S(X') \cup S(\{x_L\}) = S(X)$, but then there cannot be infinitely many antichains, so some two must be comparable, hence we are done. ■

(2) $\implies$ (1). Let $X \subseteq L$, and $Y \subseteq L$, such that $S(X) = S(Y)$. Then, $\forall y \in Y\ \exists x \in X$ such that $y \in S(X) \iff y \leq x$, hence we are done. ■

(1) $\implies$ (3). Let $(a_1, a_2, \dots)$ be an ascending chain, then it must be bounded, since otherwise let $X = \{a_i\}$. Then $\exists Y \subseteq X$ such that $|Y| < \infty$ and $S(Y) = S(\{a_i\})$. ■

$\square$

Theorem 6.7

Let L be GC-compact, $\tau_L = \{S(X) \mid |X| < \infty \land X \subseteq L\}$, and $\{S(X_i)\} \subseteq \tau_L$. Then, τ_L is a topology (not generally true if we drop the GC-compact condition). Showing that τ_L is a topology is relatively straightforward: $\bigcup S(X_i) = S(\bigcup X_i) = S(Y)$, $\bigcap \ni \alpha \implies S(\alpha) \subseteq \bigcap S(X_i)$, and $\bigcap S(X_i) = A = S(A)$.

Let $L = 2^{<\omega} = \{,0,1,00,01,10,11,\dots\}$. Then, τ_L is not closed under arbitrary unions: consider $X = \{0, 10, 110, 1110, \dots\}$, then $S(X) = L \setminus \{\underbrace{11\dots1}_{k} \mid k \geq 0\}$.

The poset must have a maximal length element, but then we can construct by taking union a longer element, so τ_L is not closed under arbitrary union, and we are done.

Claim 6.8 — $\square f$ is continuous $\forall f \in L^X$, assuming that L is GC-compact.

Proof. It suffices to show that $Y = (\square f)^{-1}(S(a))$ is closed, since f is continuous iff the pre-image of any closed set is closed.

Let $x \in Y$, then we want to show that $\exists U \ni x$ such that $U \cup Y = \emptyset$. Suppose FTSOC that $\forall W \ni x$, $W \cap Y \neq \emptyset$.

Then, $\exists w \in W \cap Y$ such that $\square f(w) \in S(a)$, hence

$$a \geq \square f(w) = \sup_{V \ni w} (\inf_{\tau \in V} f(\tau)) \geq \inf_{\tau \in W} f(\tau)$$

Thus,

$$a \geq \sup_{W \mid W \cap Y \neq \emptyset} (\inf_{\tau \in W} f(\tau)) = \square f(x)$$

But then $\square f^{-1}(a) \ni x$, contradiction. $\square$

§7 Modal Logic

$\square =$ "there is a proof of X"

$\square\square P \to \square P$.

$\square P \land \square P \leftrightarrow \square(P \land Q)$.

Exercise 7.1. Prove that $P \to \square P$ does not necessarily imply $P \to \Diamond P$.

Proof. Find a "real-world" (i.e., in a concrete system) counterexample.

We have $P \to \square P \iff \neg P \lor \square P$.

For $P \to \Diamond P$, it suffices to show $P \to \neg\square\neg P$.

$$\begin{aligned} & P \to \neg\square\neg P \\ \iff & \neg P \lor \neg\square\neg P \\ \iff & \neg(P \land \square\neg P) \end{aligned}$$

$\square$

If you add something like $\square\Diamond P = \Diamond\square P$, then $\neg\square P = \square\neg P$, like commutativity of two operations, which would probably lead to degenerate trivial cases.

Example 7.2

$f : 2^X \to 2^Y$ or like $f : 2^X \to \{0,1\}^Y$.

For example, $f(a) = \{1,2,3\}$, where $a \mapsto \{1 \to 1, 2 \to 1, 3 \to 1, 4 \to 0, 5 \to 0, \dots\} = \mathbb{1}_{\{1,2,3\}}$ which is the characteristic function of a set (returning whether an element is in the set).

There's also fuzzy sets, $f : 2^X \to [0,1]^Y$.

Example 7.3

L is a poset; $f : 2^X \to (L)^X$.

Exercise 7.4. Is it the case that $K \implies [\Diamond(P \to Q)] \to [\Diamond P \to \Diamond Q]$?

§8 Poset Modal Logic

L is a poset.

Define $\wedge$ as the "sup" of A and B.

Define $\neg$ as the "order reversing thingy".

Define $\Box P$ as "the next element in the poset after P".

One example is a tree going upwards, where stuff upwards is "stronger" in some sense.