Stokes' theorem with JCT

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Outline

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2. Things boundary
3. Differentiability classes
4. Differentiable manifolds
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6. Generalized Stokes' theorem

The typical "proof" of Stokes' theorem

First, here's the statement of Stokes' theorem:

Theorem (Stokes' theorem)

Let $A \subset \mathbb{R}^3$ be a smooth oriented surface with boundary $B = \partial A$. If $\vec{F}$ is a vector field with continuous first order partial derivatives in a region containing A, then

$$\iint_A (\nabla \times \vec{F}) \cdot \mathrm{d}A = \oint_B \vec{F} \cdot \mathrm{d}\vec{r}$$

The typical “proof” of Stokes’ theorem

Take a smooth parameterization $\vec{r} = \vec{r}(x, y)$ of A, whence A corresponds to a region R in the xy-plane, and B corresponds to the boundary $C = \partial R$. Now, express the integrals on both sides of the theorem in terms of x and y. First, convert the line integral $\oint_B \vec{F} \cdot \mathrm{d}\vec{r}$ into a line integral around C:

$$\oint_B \vec{F} \cdot \mathrm{d}\vec{r} = \oint_C \vec{F} \cdot \frac{\partial \vec{r}}{\partial x} \mathrm{d}x + \vec{F} \cdot \frac{\partial \vec{r}}{\partial y} \mathrm{d}y$$

Now, define a two-dimensional vector field $\vec{G} = (G_1, G_2)$ on the xy-plane by

$$G_1 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial x}$$

as well as

$$G_2 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial y}$$

Then, we have

$$\oint_B \vec{F} \cdot \mathrm{d}\vec{r} = \oint_C \vec{G} \cdot \mathrm{d}\vec{s}$$

where $\vec{s}$ is a position vector in the xy-plane.

Next, for the flux integral $\iint_A \nabla \times \vec{F} \cdot \mathrm{d}\vec{A}$, with respect to the parameterization,

$$\iint_A (\nabla \times \vec{F}) \cdot \mathrm{d}\vec{A} = \iint_R \nabla \times \vec{F} \cdot \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \, \mathrm{d}x \, \mathrm{d}y$$

It remains to show

$$(\nabla \times \vec{F}) \cdot \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}$$

which is not too bad to show. Hence,

$$\iint_A (\nabla \times \vec{F}) \cdot \mathrm{d}\vec{A} = \iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y$$

Indeed, we know that

$$\oint_B \vec{F} \cdot \mathrm{d}\vec{r} = \oint_C \vec{G} \cdot \mathrm{d}\vec{s}$$

Now, Green's theorem implies that

$$\oint_C \vec{G} \cdot \mathrm{d}\vec{s} = \iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y$$

Hence, we also have

$$\iint_A (\nabla \times \vec{F}) \cdot \mathrm{d}\vec{A} = \oint_B \vec{F} \cdot \mathrm{d}\vec{r}$$

as desired.

Some questions. . . .

That proof feels *okay*, but still kinda unsatisfying. What are all these handwavy terms? As mathematicians, we want more rigor! Some natural questions to consider. . . .

- Is the concept of a "boundary" well-defined?
- Is a boundary unique?
- Is the "boundary" closed?
- What is exactly "smooth"?
- How do we define integrals on a manifold?

What is a boundary?

In a topological space (X, τ), there are several equivalent definitions of a boundary ∂S of a set $S \subset X$:

- The closure of S minus the interior of S in X, that is,

  $$\partial S = \overline{S} \setminus \operatorname{int}_X S$$

  where $\overline{S} = \operatorname{cl}_X S$ is the closure of S in X and $\operatorname{int}_X S$ is the topological interior of S in X.
- The intersection of the closure of S with the closure of its complement, that is,

  $$\partial S = \overline{S} \cap \overline{X \setminus S}$$
- The set of points $p \in X$ such that every neighborhood of p contains at least one point of S and at least one point not of S.

What is a boundary?

Definition

A *manifold-with-boundary* is a manifold containing both interior points and boundary points.

Note that the boundary of an n-manifold-with-boundary is an $n-1$-manifold. (This connects to homology theory, expressing things in terms of chain complexes, which elegantly explains things such as $\operatorname{div}\operatorname{curl} = 0$ and so on.)

Jordan curve theorem

Now, we want to rigorously define the interior and exterior for $\mathbb{R}^2$ (which suffices for our purposes of proving Stokes' theorem in $\mathbb{R}^3$; we need the general Jordan–Brouwer separation theorem for the generalized Stokes' theorem for differentiable manifolds).

Definition (Jordan curve)

A *Jordan curve* or a *simple closed curve* in $\mathbb{R}^2$ is the image Γ of an injective continuous map of a circle into the plane, $\phi : S^1 \to \mathbb{R}^2$.

Definition (Jordan arc)

A *Jordan arc* in the plane is the image of an injective continuous map of a closed and bounded interval $[a, b]$ into the plane.

Note that a Jordan curve is not necessarily smooth nor algebraic.

Jordan curve theorem

Now, here's the statement of the theorem:

Theorem (Jordan curve theorem)

Let Γ *be a Jordan curve in* $\mathbb{R}^2$. *Then,* $\mathbb{R}^2 \setminus \Gamma$ *consists of exactly two connected components, one of which is bounded (the interior) and the other is unbounded (the exterior), with* Γ *being the boundary of each component.*

Outline of the proof

The Jordan curve theorem is notoriously known for its simple, intuitive, obvious statement, with a not-so-obvious proof. There are proofs that utilize homology theory, but here, we present an outline of a relatively elementary proof.

1. The Jordan curve theorem holds for every Jordan polygon Γ.
2. Every Jordan curve Γ can be approximated arbitrarily well by a Jordan polygon Γ'. (Note that this does *not* imply the statement.)
3. $\mathbb{R}^2 \setminus \Gamma$ has at least two components.
4. $\mathbb{R}^2 \setminus \Gamma$ has at most two components.

For an elementary proof, one may consult *A Proof of the Jordan Curve Theorem* by Helge Tverberg.

Differentiability classes

What actually are “smooth” functions? There are several differentiability classes to consider:

- C, that is, continuous functions.
- C^k, that is, functions with a k^{th} continuous derivative.
- C^∞, that is, infinitely differentiable functions. We call these functions *smooth*.
- C^ω, that is, functions with a convergent Taylor series expansion everywhere. We call these functions *analytic*.

Differentiability classes

Thus, the “smooth” functions we talked about before were actually just $f \in C^\infty$. Note the strict inclusion:

$$C \subsetneq C^1 \subsetneq C^2 \subsetneq \cdots \subsetneq C^k \subsetneq C^\infty$$

as well as

$$C^\omega \subsetneq C^\infty$$

For the last strict inclusion, one may consider bump functions (smooth functions with compact support), which are smooth but non-analytic.

Differentiable manifolds

Although not necessary for the $\mathbb{R}^3$ case of Stokes' theorem, we need the definition of a differentiable manifold for the proof of the generalized Stokes' theorem, which we will not prove here.

Definition

A *differentiable manifold* is a set M along with a set of injective maps $f_\alpha : U_\alpha \to M$ for open sets $U_\alpha \subset \mathbb{R}^n$, where

- $M = \bigcup_\alpha f_\alpha(U_\alpha)$
- $\forall \alpha, \beta$ with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \varnothing$, we have $f_\alpha^{-1}(W)$, $f_\beta^{-1}(W)$ both open in $\mathbb{R}^n$; also, $f_\alpha^{-1} \circ f_\beta$ and $f_\beta^{-1} \circ f_\alpha$ are differentiable.
- $\{(f_\alpha, U_\alpha)\}$, called the set of *charts*, is maximal (that is, it's the "finest" decomposition of the manifold into open sets).

Visually, a differentiable manifold is a union of these charts, which can be seen as fundamental building blocks. It's like stalks of a sheaf (kind of).

What is an integral?

Just kidding, I won't be starting from outer measures until we prove the existence of a Lebesgue measure, using Carathéodory's extension theorem. (Actually, to see this kind of systematic development of Lebesgue theory, consult any good measure theory textbook, such as *Real Analysis: Measure Theory, Integration, and Hilbert Spaces* by Stein.)

Basically, we need measure theory and exterior algebra to "properly" state integrals on a manifold. Introducing all this is *not* suitable for an $O(10)$ minute presentation, so I simply give some intuitive reasons. First, we need to "ignore" the boundary when integrating, so we need the notion of *measure zero* (negligible sets). Further, we also need to derive a measure for arbitrary manifolds, which is done in the same spirit as deriving the Lebesgue measure in $\mathbb{R}^n$.

Generalized Stokes' theorem

Lastly, although we won't prove it here, the statement for the generalized Stokes' theorem is as follows:

Theorem (Generalized Stokes' theorem)

Let M be an oriented n-dimensional manifold-with-boundary and ω be an $(n-1)$-form with compact support on M, then

$$\int_M \mathrm{d}\omega = \oint_{\partial M} \omega$$

For further reference, one may consult the following:

- *Vector Analysis* by Klaus Jänich.
- *A simple and more general approach to Stokes' theorem* by Iosif Pinelis.
- N. Bourbaki's texts — Bourbaki started as a project to rigorously prove the generalized Stokes' theorem.

Thank you!